# Dirac particles in the potential $-1/|x|$

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Since the publication of Loudon's work,[1] the problem of a particle moving under the action of the potential $-1/|x|$ (the so-called one-dimensional hydrogen atom) has been extensively considered in a number of papers in this Journal,[2–4] as well as elsewhere.[5–9] The particular interest in this potential, aside from its relevance in some physical applications,[3] comes from the fact that several problems have emerged. One of these concerns the nonexistence of bound-state solutions for the Dirac equation with the electrostatic-type potential $-1/|x|$, while the Schrödinger and the Klein–Gordon equations predict binding of particles for such a potential.[4] Here, the term electrostatic refers to the way the potential is added to the free-particle Dirac Hamiltonian. Moreover, according to the covariance of the Dirac equation, Lorentz scalar potentials can also be considered. To be specific, electrostatic and scalar potentials mean that the potential is multiplied by the same Dirac matrix as the energy and the rest mass of the particle, respectively.

A similar problem is found in the one-dimensional Coulomb potential $+|x|$. If inserted into the Schrödinger equation, all the resultant states are square-integrable and hence are bound states; the entire spectrum becomes discrete.[10] On the other hand, if added to the Dirac equation as an electrostatic potential, no bound states can occur at all[11,12]; only resonance states appear and the particle escapes to infinity by tunneling from positive to negative energy states[13] (a manifestation of the famous Klein paradox). Nevertheless, a linear scalar potential can confine relativistic particles,[14,15] and hence there exist bound states.

In this note we shall point out that the same effect occurs in the one-dimensional potential $-1/|x|$, i.e., a Lorentz scalar potential stronger than the electrostatic potential provides acceptable solutions for the bound spectrum. This scalar potential is equivalent to considering the particle mass as a function of position, an interesting possibility in the field of particle physics.[16] We also show that the particle never escapes to infinity, even if the electrostatic potential is stronger than the scalar potential so that no bound states appear; on the contrary, another interesting example of a particle falling to the origin is found.[17,18]

Let us start with the one-dimensional Dirac equation for an electrostatic potential $V(x)$ plus a Lorentz scalar potential $S(x)$

$$\left\{-i\sigma_y \frac{d}{dx} + \sigma_z[m + S(x)] - [E - V(x)]\right\}\Psi(x) = 0, \tag{1}$$

where the matrices

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

act on the two-component wavefunction $\Psi(x)$. Now we consider $V(x)$ and $S(x)$ to be the one-dimensional hydrogen atom potentials. Since the pole of the potential $-1/|x|$ behaves like an impenetrable barrier for the particles, one can consider the right $(x>0)$ and left $(x<0)$ regions as independent.[5,8] Therefore, we can take $V(x) = g_v f(x)$ and $S(x) = g_s f(x)$, where the $g$'s are the coupling constants and

$$f(x) = \begin{matrix} -1/x, & x>0, \\ +\infty, & x<0. \end{matrix}$$

This choice of the potential function, which has also been considered by Nieto[19] for nonrelativistic particles, leads to the following first-order coupled equations:

$$\frac{d}{dx}\Psi_u(x) = +\left(E + m + \frac{g_v - g_s}{x}\right)\Psi_l(x),$$
$$\frac{d}{dx}\Psi_l(x) = -\left(E - m + \frac{g_v + g_s}{x}\right)\Psi_u(x), \tag{2}$$

for the upper $\Psi_u$ and lower $\Psi_l$ components of $\Psi$. Fortunately, Eq. (2) has exactly the same form as the radial Dirac equation for the spherical Coulomb potential $V(r) = -g_v/r$ and $S(r) = -g_s/r$ [compare Eq. (2) with (3.86) of Ref. 20], except that there exist no spin terms in one dimension. To pass from the three-dimensional to the one-dimensional equation, we may carry out the symbolic substitution $j \to -\frac{1}{2}$, where $j$ denotes the angular momentum. We should stress that this is only a formal prescription without any physical meaning. Therefore, following the steps of Ref. 20, the (unnormalized) solutions of Eq. (2) are readily found in terms of confluent hypergeometric functions

$$\begin{pmatrix}\Psi_u(x) \\ \Psi_l(x)\end{pmatrix} = x^{\nu}e^{-\eta x}(m \pm E)^{1/2}[\pm(g_v m + Eg_s)$$
$$\times F(-n, 1+2\nu; 2\eta x) + (\nu\eta - g_v E - mg_s)$$
$$\times F(1-n, 1+2\nu; 2\eta x)], \tag{3}$$

where the upper (lower) sign refers to the upper (lower) component. Here, $n$ denotes a positive integer, $\nu \equiv +(g_s^2 - g_v^2)^{1/2}$, and $\eta \equiv (m^2 - E^2)^{1/2}$ is a real parameter for bound states $(m > |E|)$. The energy levels are obtained through the quantum condition

$$(g_v E_n + mg_s)/(m^2 - E_n^2)^{1/2} = n + (g_s^2 - g_v^2)^{1/2}. \tag{4}$$

For the sake of simplicity, we introduce the quantity

$$c(x) \equiv \Psi^+(x)[V(x) + \sigma_z S(x)]\Psi(x)$$
$$= (g_v + g_s)|\Psi_u(x)|^2 f(x)$$
$$+ (g_v - g_s)|\Psi_l(x)|^2 f(x), \tag{5}$$

so the expectation value of the potential energy is then given by

$$\int_0^\infty c(x)dx.$$

This integral should remain finite in order to obtain acceptable bound-state solutions.

Let us now consider some special cases.

(a) Pure electrostatic potential $(g_v \neq 0, g_s = 0)$. In this particular situation $\nu = i|g_v|$ is an imaginary parameter. The wavefunction presents an essential singularity and oscillates rapidly near the origin $\Psi(x) \sim \cos(|g_v|\log x + \text{const})$ without reaching any limit, and hence it possesses a continuous spectrum for all energy values because no adequate boundary condition is found at $x \to 0$. Therefore, there exist no bound solutions, no matter how large $g_v$ is.[17] Since the particle cannot escape to infinity for $|E| < m$ [as can be checked from the asymptotic behavior of Eq. (2)], the particle escapes to the origin and "falls" to the center: The one-dimensional Dirac hydrogen atom would be absolutely unstable. Also note from Eq. (4) that the particle energy develops an imaginary part, which is a general feature of problems where collapse to the center occurs. This imaginary part is related to the probability of pair production by the overcritical electrostatic field.[21] Moreover, in view of Eq. (5), $c(x)$ behaves like $\sim x^{2i|g_v|-1}$ as $x$ goes down to zero, and consequently the expectation value of the potential energy becomes infinite. In a similar fashion, the falling of the particle to the center in the three-dimensional Coulomb potential may occur, when $Z\alpha > j + \frac{1}{2}$. A proper treatment of Dirac particles in overcritical electrostatic potentials has been given elsewhere.[21-23]

The same conclusions hold for a nonvanishing scalar coupling constant, whatever $|g_v| > |g_s|$: No bound-state solutions can be obtained at all.

(b) Pure scalar potential ($g_s \neq 0$, $g_v = 0$). Now $\nu$ becomes a positive real parameter and hence the wavefunction behaves correctly near the origin. In addition, $c(x) \sim x^{2|g_s|}$, leading to a finite expectation value of the potential energy. Also, bound states can occur even if $g_v$ has a nonzero value, smaller than $g_s$. Energy levels appear in pairs; from Eq. (4) we have

$$E_n/m = \pm [1 - g_s^2/(n + g_s^2)]^{1/2} \tag{6}$$

for positive $g_s$ (attractive scalar potential), while there is no binding of particles for negative $g_s$ (repulsive potential). The states with positive (negative) energy correspond to particles (antiparticles) reflecting that the scalar potential binds particles as well as antiparticles.

(c) Equally mixed potential ($g_s = g_v$ and $\nu = 0$).

This particular choice of the coupling constants leads to a Schrödingerlike equation for the $\Psi_u(x)$, as can be seen from Eq. (2). Expanding $\Psi_u(x)$ in powers of $x$, from expression (3), one obtains $\Psi_u(x) \sim x$, so that $c(x) \sim x$ near the origin. Therefore, the wavefunction is square-integrable and the expectation value of the potential energy remains finite, so $\Psi$ represents truly bound states. The energy levels are given by

$$E_n/m = 1 - 2g_v^2/(g_v^2 + n^2). \tag{7}$$

For weak coupling we have $(E_n - m) \simeq -m(2g_v)^2/2n$, resembling Balmer's formula (the factor of 2 in the coupling constant is due to the two equal terms of the potential).

In conclusion, the one-dimensional potential $-1/|x|$ can bind Dirac particles only if considered as a Lorentz scalar potential, while no binding of particles can occur for electrostatic potentials. This means that, unlike the three-dimensional electrostatic potential $-g_v/r$, the potential $-g_v/|x|$ becomes overcritical[22] even if $g_v$ is small, and then the particle "falls" to the center. Vacuum polarization, or some other way to regularize the potential near the origin, should prevent the collapse of the particle into the center. Finally, we have found that an equal mixture of both potentials can bind particles.


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