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Spectroscopy of a Perturbed Dirac Oscillator.

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Abstract. – Energy levels of the Dirac oscillator perturbed by arbitrary nonlocal separable vector and scalar potentials are found in a closed form. Unlike the nonrelativistic case, the perturbation potential radically changes all energy levels of the oscillator. The δ-function limit of the perturbation potential is discussed in detail.

Avakian *et al.* [1] have analysed the effects of a δ-shaped potential on the nonrelativistic harmonic-oscillator spectroscopy. These authors found that the δ-function potential only affects even harmonic-oscillator levels. Also they demonstrated that, in some limiting cases, the existence of this singular potential leads to an anomalous double degenerate levels and to the falling of the particle into the centre. This problem is of interest in quark physics since the effects of Coulomb-like potentials, which could appear between quarks [2], may be qualitatively described replacing the actual r^{-1} dependence by a more simplified short-ranged function. In this sense, the one-dimensional δ-function potential is a good analog for the Coulomb interaction. Relativistic effects on the spectroscopy of the singular harmonic-oscillator have been considered by Domínguez-Adame and Maciá [3]. These authors studied the Dirac equation with an equality mixed vector (the time component of a Lorentz vector) and scalar harmonic-oscillator potentials plus a Lorentz scalar δ-shaped potential. The resulting equation reduces to a Schrödinger-like form for the upper component of the wavefunction, so its solution is readily found.

Moshinsky and Szczepaniak [4] have recently considered a very interesting linear interaction in the Dirac equation, which reduces to a standard harmonic-oscillator equation with a strong spin-orbit coupling in the nonrelativistic limit. These authors gave to this problem the name of Dirac oscillator. Moreno and Zentella [5] have suggested that the Dirac oscillator is a good candidate to explain the observed confinement of quarks. The aim of this letter is to study the spectroscopy of the Dirac oscillator in a $(1+1)$-dimensional space perturbed by an arbitrary potential. Therefore, we present a generalization of previous works [1, 2], in the sense that we not only consider a relativistic equation but also we do not restrict ourselves to a δ-function perturbation potential. The perturbation potential will be replaced by a nonlocal separable potential, as defined by Calkin *et al.* [6]. In fact, this is not a limitation since it is always possible to find a nonlocal separable potential (or a sum of

nonlocal potentials) which reproduces any set of given wave functions [7]. This procedure enables us to obtain an exactly solvable equation, leading to closed forms for the energy levels of the perturbed oscillator.

The Dirac oscillator Hamiltonian H_{DO} in one-space dimensions (say x) can be obtained from the free-particle Hamiltonian $H_0 = \alpha p + \beta m$ in a nonminimal way replacing m by $m + \imath m\omega\alpha x$ [8]. Here ω denotes the oscillator frequency and α, β are 2×2 Hermitian, traceless matrices with square unity such that $\alpha\beta + \beta\alpha = 0$. Therefore $H_{\mathrm{DO}} = \alpha(p - \imath m\omega\beta x) + \beta m$, so the perturbed Dirac oscillator equation for steady states reads

$$[\alpha(p - \imath m\omega\beta x) + \beta m - E + U(x)]\psi(x) = 0\,. \tag{1}$$

The perturbation potential is taken to be $U(x) = V(x) + \beta S(x)$, where V and S are the vector and scalar parts of the potential.

Solutions of the perturbed Dirac oscillator equation (1) may be expressed in terms of the Green's function for the Dirac oscillator $G(x, x'; E)$ as

$$\psi(x) = -\int \mathrm{d}x'\, G(x, x'; E)\, U(x')\psi(x')\,. \tag{2}$$

The Green's function is a 2×2 matrix-valued function satisfying the inhomogeneous differential equation

$$[\alpha(p - \imath m\omega\beta x) + \beta m - E]\, G(x, x'; E) = I_2\,\delta(x - x') \tag{3}$$

subject to suitable boundary conditions. Here I_2 stands for the 2×2 unit matrix. Thus we explicitly write

$$G(x, x'; E) = \begin{pmatrix} G_{++}(x, x') & G_{+-}(x, x') \\ G_{-+}(x, x') & G_{--}(x, x') \end{pmatrix}, \tag{4}$$

where the dependence on E of the matrix elements is understood. For the sake of simplicity we define the dimensionless variable $\xi = (2m\omega)^{1/2} x$ and

$$g_{\pm}(\xi, \xi') = \frac{(2m\omega)^{1/2}}{E \pm m} G_{\pm\pm}(\xi, \xi')\,, \tag{5a}$$

$$\lambda_{\pm} = \frac{E^2 - m^2}{2m\omega} \pm \frac{1}{2} - \frac{1}{2}\,. \tag{5b}$$

Inserting (4) in eq. (3), we obtain four coupled differential equations, which are easily decoupled in the standard fashion. Taking the representation $\alpha = \sigma_x$ and $\beta = \sigma_z$ (σ's denote the Pauli matrices), we have

$$G_{\pm\mp}(\xi, \xi') = -\imath\left(\frac{\partial}{\partial\xi} \mp \frac{\xi}{2}\right) g_{\pm}(\xi, \xi') \tag{6}$$

and

$$\left(-\frac{\partial^2}{\partial\xi^2} + \frac{\xi^2}{4} - \lambda_{\pm} - \frac{1}{2}\right) g_{\pm}(\xi, \xi') = \delta(\xi, \xi')\,. \tag{7}$$

Notice that $g_{\pm}(\xi, \xi')$ are directly related to the Green's function of the standard one-dimensional harmonic oscillator, provided that $g_{\pm}(\xi, \xi')$ vanish as $|\xi| \to \infty$. Equation (7) is

solved by Sturm-Liouville theory, and solutions may be written in terms of parabolic cylinder functions $D(\lambda, \xi)$ as

$$g_{\pm}(\xi, \xi') = \frac{D(\lambda_{\pm}, -\xi_{<})\, D(\lambda_{\pm}, \xi_{>})}{2D(\lambda_{\pm}, 0)\, D(\lambda_{\pm}+1, 0)}, \tag{8}$$

where $\xi_{<} = \min(\xi, \xi')$ and $\xi_{>} = \max(\xi, \xi')$. The relation $D(\lambda, 0) = 2^{\lambda/2}\Gamma(1/2)\Gamma(1/2 - \lambda/2)$ is useful in order to compute the denominator. The complete Green's function of the Dirac oscillator is determined with the aid of (5) and (6). It is worth mentioning that $G_{\pm\pm}(\xi, \xi')$ are continuous functions in the plane (ξ, ξ'), whereas $G_{\pm\mp}(\xi, \xi')$ exhibit a finite jump on the line $\xi = \xi'$.

As we mentioned above, the actual local potential $U(x)$ in (1) is replaced by a projective operator V_{PO} of the form

$$V_{\mathrm{PO}}\psi = (g_{\mathrm{v}} + \beta g_{\mathrm{s}})\, v(x) \int \mathrm{d}x'\, v(x')\, \psi(x'), \tag{9}$$

where g_{v} and g_{s} are the vector and scalar coupling constants, respectively, and $v(x)$ is a shape function. We take even functions $v(x) = v(-x)$ hereafter. Different shape functions for the vector and scalar parts of the potential could also be handled, although we omit here this case without loosing generality. From (2) one gets

$$\psi(x) = -\int \mathrm{d}x'\, G(x, x'; E)\, v(x')(g_{\mathrm{v}} + \beta g_{\mathrm{s}})\, \chi, \tag{10}$$

where

$$\chi = \int \mathrm{d}x\, v(x)\, \psi(x). \tag{11}$$

The consistency of eqs. (10) and (11) leads to

$$\mathrm{Det}\left(1 + \int \mathrm{d}x \int \mathrm{d}x'\, G(x, x'; E)\, v(x)\, v(x')(g_{\mathrm{v}} + \beta g_{\mathrm{s}})\right) = 0. \tag{12}$$

Since $v(x)$ is an even function of the spatial coordinate, it is an easy matter to demonstrate that integrals involving off-diagonal matrix elements of $G(x, x'; E)$ vanish, according to (6) and (8). Therefore, the consistency equation reads

$$[1 + (g_{\mathrm{v}} + g_{\mathrm{s}})J_{+}][1 + (g_{\mathrm{v}} - g_{\mathrm{s}})J_{-}] = 0 \tag{13}$$

with the notation

$$J_{\pm} = \frac{E \pm m}{\sqrt{2m\omega}} \int \mathrm{d}\xi \int \mathrm{d}\xi'\, g_{\pm}(\xi, \xi')\, v(\xi)\, v(\xi'). \tag{14}$$

Equation (13) determines the energy levels of the perturbed Dirac oscillator. These levels will depend, as expected, on the exact form of the shape function and on the values of the coupling constants. Both the shape function and the coupling constants can be calculated from the solutions of the wave equation for the potential $U(x)$ alone [7]. In general situations the procedure may be rather involved although straightforward. In order to circumvent this problem, one could try different simple shape functions until the desired precision of the

results is reached. The only limitation comes from the fact that integrals appearing in (14) must converge to obtain acceptable solutions. We should stress that we are not restricted to zero-ranged potentials, as in previous models of perturbed relativistic oscillators [3].

As an example of the treatment introduced above, we take the limit $v(x) \to \delta(x)$. This case provides a direct comparison with the spectroscopy of the nonrelativistic singular oscillator, previously considered by Avakian *et al.* [1]. Defining the function

$$\eta(E) = \frac{E+m}{2\sqrt{m\omega}} \frac{\Gamma(-(E^2-m^2)/4m\omega)}{\Gamma(1/2-(E^2-m^2)/4m\omega)}, \tag{15}$$

we find from (13) that

$$1 + (g_s^2 - g_v^2)/4 + \frac{g_s}{2}\left[\eta(E) + \frac{1}{\eta(E)}\right] + \frac{g_v}{2}\left[\eta(E) - \frac{1}{\eta(E)}\right] = 0. \tag{16}$$

Real solutions of this equation yield the energy levels of the singular Dirac oscillator. Figure 1 shows these levels as a function of the coupling constants for the case $\omega = m$. We can draw several conclusions from the above results.

a) In the nonrelativistic limit and for weak coupling (g_v^2, g_s^2, ω/m and E/m much smaller than unity), eq. (16) leads to the same spacing between levels as that quoted by Avakian *et al.* [1] in dealing with the Schrödinger equation, although energy levels are shifted downwards by a constant factor $\omega/2$. This is a peculiarity of the Dirac oscillator with regard to its nonrelativistic limit [9].

b) When the singular potentials are adiabatically turned off ($g_v, g_s \to 0$), we obtain the energy spectrum $E = +m$ and $E^2 = m^2 + 2nm\omega$, n being a positive integer (recall that Γ function becomes divergent for zero or negative integer arguments). This spectrum is in agreement to that obtained by Domínguez-Adame and Méndez [9] solving the unperturbed Dirac oscillator equation in (1 + 1)-dimensions. On the contrary, taking into account that $\Gamma(-x)/\Gamma(1/2-x) \to (-x)^{-1/2}$ as $x \to \infty$, we find that energy levels are given by

$$[2(m^2-E^2)^{1/2} + (m+E)(g_s+g_v)][2(m^2-E^2)^{1/2} + (m-E)(g_s-g_v)] \tag{17}$$

as ω vanishes (*i.e.* neglecting the effects of the Dirac oscillator interaction). This result has

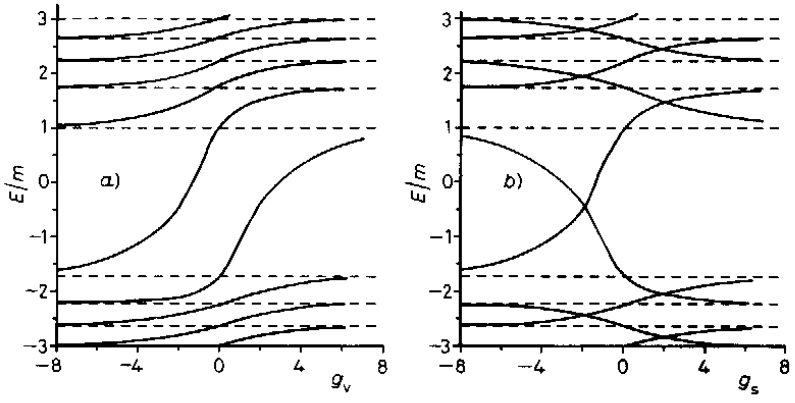

Fig. 1. – Energy levels of the Dirac oscillator with $\omega = m$ perturbed by *a*) vector and *b*) scalar, nonlocal δ-function potentials. Dashed lines indicate unperturbed levels.

been previously found by Calkin *et al.* [6] in dealing with the nonlocal separable δ-function potential in the Dirac equation.

c) The particle energy remains finite even if the singular potentials are rather strong. Therefore, unlike the nonrelativistic case [1], the particle cannot fall into the centre.

d) From fig. 1 we notice that the singular Dirac oscillator can bind particles as well as antiparticles. This is in contrast to the relativistic singular harmonic-oscillator previously introduced by Domínguez-Adame and Maciá [3], for which antiparticle states retain their unbound continuum and only one negative-energy bound state appears for attractive δ-function potentials.

e) For pure-vector, singular potentials ($g_v \neq 0$, $g_s = 0$), we observe in fig. 1 that each energy level is an increasing function of the coupling constant g_v. Energy levels for the limit cases $g_v \to \infty$ and $g_v \to -\infty$ are just the same and coincide with those of the unperturbed Dirac oscillator. This result is explained by noting that the transmission coefficient for vector nonlocal δ-function potential, calculated in [6]

$$T = \frac{(E^2 - m^2)(1 + g_v^2/4)^2}{(E^2 - m^2)(1 - g_v^2/4)^2 + E^2 g_v^2} \tag{18}$$

tends to unity as $|g_v| \to \infty$, no matter the value of E. Therefore the nonlocal δ-function potential becomes transparent to all energy and presents no effects on the Dirac oscillator levels.

f) For pure-scalar, singular potentials ($g_v = 0$, $g_s \neq 0$), energy levels are alternatively shifted upwards and downwards from the unperturbed ($g_s \to 0$) levels. The same behaviour is observed in dealing with scalar, local δ-function potentials [3]. As discussed in the previous case, we find the same result again for the limit situation $|g_s| \to \infty$. The transmission coefficient for scalar nonlocal δ-function potential is [6]

$$T = \frac{(E^2 - m^2)(1 - g_s^2/4)^2}{(E^2 - m^2)(1 + g_s^2/4)^2 + m^2 g_s^2}, \tag{19}$$

which goes to unity in that limiting case. Also note that the transmission coefficient goes down to zero for $|g_s| = 2$. Hence, a particle moving in the left (right) region cannot go through the barrier and will remain there indefinitely. Actually, one has to deal with two separate potentials, so an apparent double degeneracy of levels occurs. This degeneracy explains the crossing of levels for $|g_s| = 2$, as seen in fig. 1.

In summary, we have solved the perturbed Dirac oscillator equation by means of the Green's function for the unperturbed equation. The perturbation potential has been replaced by a nonlocal separable interaction, thus leading to closed forms for the energy levels. The δ-function limit has been discussed in detail.

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