EUROPHYSICS LETTERS 15 April 1989
Europhys. Lett., 8 (8), pp. 711-715 (1989)

On Relativistic Singular Harmonic-Oscillator Potentials.

F. DOMÍNGUEZ-ADAME and E. MACIÁ (*)
Departamento di Física de Materiales, Facultad de Físicas
Universidad Complutense - 28040 Madrid, Spain

(received 11 July 1988; accepted in final form 17 February 1989)

PACS. 01.55 – General physics.
PACS. 03.65 – Quantum theory; quantum mechanics.

Abstract. – The Dirac equation for a harmonic-oscillator potential plus a Lorentz scalar delta-shaped term has been solved. This singular potential changes the usual oscillator energy levels. Unlike the nonrelativistic treatment, the particle cannot fall into the centre for any negative value of the delta-function strength. The apparent occurrence of double degenerate energy levels for some limiting cases is found.

A large number of papers have appeared in the literature about the harmonic-oscillator quark model. The nonrelativistic treatment of this model explains the ground state of baryons and mesons in a rather simple way. Moreover, confining properties of quarks are well understood within this framework. Nevertheless, one can expect that relativistic treatments should give better results. In this way, the Dirac equation has been solved for an equally mixed harmonic potential formed by adding a scalar potential to the usual time component vector potential [1]. This scalar term is required for obtaining bound-state solutions of the Dirac equation, since harmonic vector coupling cannot bind particles due to the Klein paradox. This quark model has properties quite similar to the MIT bag model [2].

Concerning quark physics at small distances, Avakian *et al.* [3] have analysed the effects of a delta-shaped potential on the nonrelativistic harmonic-oscillator spectroscopy. They have found that a δ-function potential changes radically the even harmonic-oscillator levels, while odd levels remain unchanged and, in some limiting cases, leads to the production of anomalous double degenerate energy levels and to the falling of the particle into the centre. Statistical properties of this approach have been considered in a recent paper by Janke *et al.* [4]. Nevertheless, relativistic effects have not been taken into account in the above-mentioned works.

The aim of the present letter is to generalize the previous model to be applied for Dirac particles. We consider an equally mixed harmonic potential plus a Lorentz scalar delta-shaped term, *i.e.* a potential of the form $V(x) = (1/4)(1 + \sigma_z) m\omega^2 x^2 + \Omega\sigma_z \delta(x)$, Ω being the strength of the δ-function potential, and σ's denote the usual 2×2 Pauli matrices. In order to

(*) On leave from Agrupación Astronómica de Sabadell, Barcelona (Spain).

solve the Dirac equation for this potential, we should use appropriate boundary conditions around the origin. As Sutherland *et al.* [5] pointed out, the concept of a relativistic δ-function potential exhibits some ambiguities. A reasonable criterion to resolve these ambiguities has been given by McKellar *et al.* [6]. For Lorentz scalar potentials that approach a δ-function limit (located to x_0), they have found the following boundary condition for the two-component wave function of the Dirac particle (in the standard representation)

$$\psi(x_0^+) = \begin{pmatrix} \cosh(\Omega/\hbar c) & i\sinh(\Omega/\hbar c) \\ -i\sinh(\Omega/\hbar c) & \cosh(\Omega/\hbar c) \end{pmatrix} \psi(x_0^-). \tag{1}$$

First, for providing some insight into the general problem, we shall apply this result to study bound states of the one-dimensional Dirac equation [7]

$$\left\{-i\hbar c\sigma_x \frac{\mathrm{d}}{\mathrm{d}x} + \sigma_z mc^2 + V(x) - E\right\}\begin{pmatrix} \phi(x) \\ \chi(x) \end{pmatrix} = 0, \tag{2}$$

for an attractive Lorentz scalar potential $V(x) = \Omega\sigma_z f(x)$ ($\Omega < 0$) in the limit $f(x) \to \delta(x)$. $\phi(x)$ and $\chi(x)$ denote the upper and the lower components on the wave function $\psi(x)$, respectively. For $x \neq 0$ we have

$$\chi'(x) = -i\eta\xi^{-1}\phi(x), \tag{3a}$$

$$\phi'(x) = i\eta\xi\chi(x), \tag{3b}$$

where the prime indicates differentiation with respect to the argument. $\xi \equiv (mc^2 + E)/(mc^2 - E)^{1/2}$ and $\eta\hbar c \equiv (m^2c^4 - E^2)^{1/2}$ are real parameters for bound states. The solution of eqs. (3) is readily found

$$\psi(x) = \begin{cases} A\begin{pmatrix} 1 \\ i\xi^{-1} \end{pmatrix} \exp[-\eta x], & x > 0, \\ B\begin{pmatrix} 1 \\ -i\xi^{-1} \end{pmatrix} \exp[\eta x], & x < 0, \end{cases} \tag{4}$$

and using the boundary condition (1) we can obtain the particle energy

$$-\operatorname{tgh}(\Omega/\hbar c) = +\sqrt{1-\varepsilon^2} \equiv f(\varepsilon), \tag{5}$$

with $\varepsilon \equiv E/mc^2$. Note that no bound states can occur for positive values of the potential strength. Figure 1 shows the graphical solution of this equation for an arbitrary negative value of Ω. Two values (positive and negative) of the energy are possible for each value of Ω, so this potential can bind particles and antiparticles alike. Note that the particle energy remains finite and never crosses zero even if the potential becomes very strong. Therefore, the particle cannot fall into the centre, as occurs in the nonrelativistic case.

Now, bearing in mind the results obtained above, we are faced with the solution of the Dirac equation (2) for a singular harmonic-oscillator potential. Thus we write

$$-i\hbar c\chi'(x) + \left(mc^2 - E + \frac{1}{2}m\omega^2x^2 + \Omega\delta(x)\right)\phi(x) = 0, \tag{6a}$$

$$-i\hbar c\phi'(x) + (-mc^2 - E - \Omega\delta(x))\chi(x) = 0. \tag{6b}$$

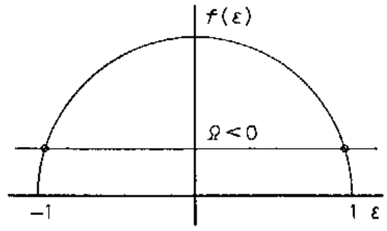

Fig. 1. – Bound states energy for a single attractive δ-function potential.

For the sake of simplicity, we introduce the notation $z \equiv [2(1+\varepsilon)\, m^2\omega^2/\hbar^2]^{1/4} x$, $\lambda + 1/2 \equiv mc^2(\varepsilon - 1)[(1+\varepsilon)/2]^{1/2}/\hbar$ and $\alpha \equiv (\hbar\omega/2mc^2)^{1/2}[8/(1+\varepsilon)^3]^{1/4}$, so that

$$i\chi'(z) + \alpha\left(\lambda + \frac{1}{2} - \frac{1}{4}z^2\right)\phi(z) = 0, \tag{7a}$$

$$i\alpha\phi'(z) = -\chi(z). \tag{7b}$$

Since $|\varepsilon| \leq 1$, λ becomes a real parameter. Inserting (7*b*) in (7*a*) we obtain

$$\phi''(z) + \left(\lambda + \frac{1}{2} - \frac{1}{4}z^2\right)\phi(z) = 0. \tag{8}$$

Solutions of this differential equation are expressed in terms of parabolic cylinder functions $D_\lambda(z)$. Since $\phi(z)$ must vanish as $|z| \to \infty$, and after using eq. (7*b*), we can readily write the particle wave function

$$\psi(x) = \begin{cases} A\begin{pmatrix} D_\lambda(z) \\ -i\alpha D'_\lambda(z)\end{pmatrix}, & z > 0, \\ B\begin{pmatrix} D_\lambda(-z) \\ -i\alpha D'_\lambda(-z)\end{pmatrix}, & z < 0, \end{cases} \tag{9}$$

where A and B are constant. The recurrence relations

$$D'_\lambda(z) = -(z/2)\,D_\lambda(z) + \lambda D_{\lambda-1}(z) = (z/2)\,D_\lambda(z)\,D_{\lambda+1}(z)$$

are useful for obtaining $D'_\lambda(\pm z)$.

Using the boundary condition (1) and the relation $D_\lambda(0) = 2^{\lambda/2}\Gamma(1/2)/\Gamma(1/2 - \lambda/2)$, we obtain two algebraic equations for A and B. Requiring the determinant to vanish for nontrivial solutions, we find

$$-\mathrm{tgh}\,(\Omega/\hbar c) = 2\left[\left(1/\sqrt{2}\,\alpha\right)\Gamma\left(-\frac{\lambda}{2}\right)\Big/\Gamma\left(\frac{1}{2} - \frac{\lambda}{2}\right) + \sqrt{2}\,\alpha\,\Gamma\left(\frac{1}{2} - \frac{\lambda}{2}\right)\Big/\Gamma\left(-\frac{\lambda}{2}\right)\right]^{-1}. \tag{10}$$

This transcendental equation determines the energy levels and has to be solved numerically. For weak coupling, the nonrelativistic limit is found to be

$$-(\Omega/\hbar c)(2m/\hbar\omega)^{1/2} = 2\sqrt{2}\,\Gamma\left(\frac{1}{2} - \frac{\lambda}{2}\right)\Big/\Gamma\left(-\frac{\lambda}{2}\right), \tag{11}$$

with $\lambda + 1/2 \to (E - mc^2)/\hbar\omega$, in agreement with the results of [3] and [4]. In the particular case $\Omega \to 0$ we have $\Gamma(1/2 - \lambda/2) = \infty$ or $\Gamma(-\lambda/2) = \infty$ from eq. (10), leading to the quantum condition $\lambda = n$ (n being a nonnegative integer) and therefore the particle energy is given by

$$(E - mc^2)(E + mc^2)^{1/2} = \hbar\omega(2mc^2)^{1/2}(n + 1/2). \tag{12}$$

The first few energy levels for $\Omega = 0$ are shown in table I (as an example, we shall take $\hbar\omega = mc^2$ in our numerical results). This particular situation has been previously solved by Ravndal [1]. This author has considered a three-dimensional model of the harmonic coupling, so the factor 1/2 of the eq. (12) must be replaced by a factor 3/2.

TABLE I. – *Energy levels for a relativistic harmonic-oscillator ($\hbar\omega = mc^2$) potential without delta-shaped term ($\Omega = 0$).*

n	0	1	2	3	4	5	6
E/mc^2	1.45	2.19	2.81	3.37	3.88	4.36	4.81

The graphical solution of eq. (10) is plotted in fig. 2. The right-hand side of this equation is denoted by $f(\varepsilon)$, whose zeros ($\Omega = 0$) are given in table I. The upper (lower) half-plane corresponds to negative (positive) values of the potential strength. Unlike the nonrelativistic treatment, the presence of a delta-shaped potential changes every energy levels of the relativistic harmonic oscillator. For a nonzero value of Ω, the energy levels are

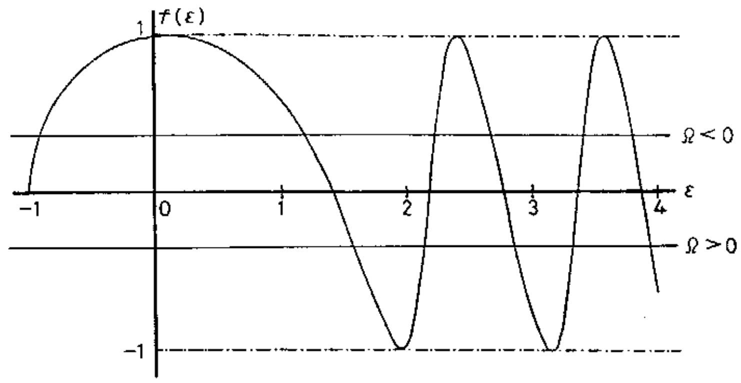

Fig. 2. – Graphical determination of the energy levels for a relativistic harmonic-oscillator ($\hbar\omega = mc^2$) potential with a δ-function potential term.

alternatively shifted upward and downward, as seen in fig. 2. Note the occurrence of a negative energy level for negative values of Ω, which seems to be related to the above-mentioned energy spectrum of a single attractive δ-function potential (for energy values ranging from $-mc^2$ to $+mc^2$, fig. 2 is quite similar to fig. 1). The harmonic-oscillator potential does not act on the antiparticle states, which retain their unbound continuum spectrum; therefore, negative-energy bound states appearing when the scalar δ-function potential is introduced represent binding of antiparticles, just as occurs in the case of the pure delta-shaped attractive potential.

Finally, we should remark the production of double degenerate levels as $|\Omega| \to \infty$ (the first few degenerate levels are shown in table II), which is a common feature to the

TABLE II. – *Double degenerate energy levels for a relativistic harmonic-oscillator* ($\hbar\omega = mc^2$) *potential with an infinite delta-shaped term* ($|\Omega| = \infty$).

	Energy levels (E/mc^2)		
$\Omega = -\infty$	0.08	2.43	3.59
$\Omega = +\infty$	1.95	3.15	4.13

nonrelativistic treatment given by Avakian [3]. Nevertheless, the level distribution in these limiting spectra depend on the repulsive or attractive character of the δ-function potential, while nonrelativistic levels [3] do not. The occurrence of these degenerate levels is easily understood since a strong scalar δ-function potential behaves like an impenetrable wall, no matter the sign of the strength (the transmission coefficient vanishes [6]). Therefore, a particle moving in the left (right) region cannot go through the barrier into the right (left) region. Actually, one has to deal with two separate potentials, so the theorem of nondegeneracy in one dimension is fulfilled.

We have proved that the wave function of a Dirac particle in a singular harmonic-oscillator potential can be explicitly written down. Energy levels of a relativistic harmonic-oscillator are radically changed by adding a Lorentz scalar delta-shaped term. This result is expected to be interesting in quark physics at small distances. The particle cannot fall into the centre for any negative value of the delta-function strength. The apparent occurrence of double degenerate levels as the delta-function potential (attractive or repulsive) becomes large has also been shown.

* * *

The authors thank Dr. M. T. PÉREZ-FRÍAS and P. FERNÁNDEZ for the critical reading of the manuscript.

REFERENCES

[1] RAVNDAL F., *Phys. Lett. B*, **113** (1982) 57.
[2] DE GRAND T., JAFFE R. L., JOHNSON K. and KISKIS J., *Phys. Rev. D*, **12** (1975) 2060.
[3] AVAKIAN M. P., POGOSYAN G. S., SISSAKIAN A. N. and TER-ANTONYAN V. M., *Phys. Lett. A*, **124** (1987) 233.
[4] JANKE W. and CHENG B. K., *Phys. Lett. A*, **129** (1988) 140.
[5] SUTHERLAND B. and MATTIS D. C., *Phys. Rev. A*, **24** (1981) 1194.
[6] MCKELLAR B. H. J. and STEPHENSON G. J. S., *Phys. Rev. A*, **36** (1987) 2566; *Phys. Rev. C*, **35** (1987) 2262.
[7] STÉSLICKA M. and DAVIDSON S. G., *Phys. Rev. B*, **1** (1970) 1858.