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Localized solutions of one-dimensional nonlinear Dirac equations with point interaction potentials

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Abstract. We find the exact localized solutions of a class of nonlinear Dirac equations perturbed by a point interaction potential, that is, any sharply peaked potential approaching the δ-function limit. A detailed analysis of the existence conditions of these localized solutions is carried out. We show that our results agree with non-relativistic predictions when both the self-coupling and the point interaction potential are weak. Lower bounds to the size of the localized solutions are presented.

One of the most important questions about soliton-type waves is their stability under the influence of a general perturbation. For nonlinear scalar fields, the stability problem has been completely solved by Shatah and Strauss [1, 2]. The Shatah–Strauss formalism has been extended to study the stability of spinor fields satisfying the nonlinear Dirac equation (NLDE) under special perturbations [3, 4], namely dilations and amplitude transformations. In addition, extensive numerical work has been devoted to this problem [5–7].

Up to now, however, the behaviour of localized solutions of the NLDE under the influence of external potentials remains somewhat unexplored. A basic question is to discover whether the potential can destroy these localized solutions. The aim of this work is to provide a first study towards clarifying this point. We consider the NLDE with vector self-interaction plus a short-ranged electrostatic-like potential (the timelike component of a Lorentz 4-vector) in one space dimension. There are no exact solutions in three dimensions even for the unperturbed NLDE, so that analytical solutions in one dimension assume an additional importance. The situation we are dealing with is equivalent to a *modified* Thirring model [8], in which the influence of a external potential is included. In order to obtain a solvable model, we will replace the actual external potential by a point interaction potential, i.e. any sharply peaked potential which approachs the δ-function limit (a more rigorous definition can be found in [9]). Within limitations—some of them will be discussed below—the δ-function potential is a good candidate to replace more structured and more complex short-ranged potentials [10].

Hence we aim to find the localized solutions and the existence conditions of these solutions of the following NLDE (we use units such that $\hbar = c = 1$):

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(x,t) = \left(-\mathrm{i}\alpha\frac{\partial}{\partial x} + \beta m + \lambda v(x) - 2g\Psi^{\dagger}\Psi\right)\Psi(x,t) \qquad (1)$$

where the exact form of the external potential $\lambda v(x)$ will be specified later, λ being the coupling constant and $v(x)$ a shape function. Without losing validity, we assume a self-interaction with $0 < g < \pi$. α and β are 2×2 Hermitian and anticommuting, traceless matrices with square unity, acting upon the two-component wavefunction Ψ.

As was pointed out by Sutherland and Mattis [11], some ambiguities appear taking the limit $v(x) \to \delta(x)$ at the outset in the linear Dirac equation ($g = 0$), because potentials of different shapes which approach the δ-function limit (zero width and constant area) give wavefunctions reaching different values at the discontinuity point. The origin of these ambiguities was clarified by McKellar and Stephenson [12], and it is related to the fact that the Dirac equation is linear rather than quadratic in momentum. Due to this linearity in momentum, the wavefunction itself must be discontinous at $x = 0$ in order to account the singularity of the potential term. However, the product $\delta(x)\theta(x)$, where θ is the step function, is not well defined in a strict distribution-theory sense, so there exists arbitrariness regarding the definition of point interaction potentials in the linear equation. Moreover, it is quite clear that the same anomaly will occur in the NLDE.

The first task then is to define a point interaction potential for the NLDE avoiding these difficulties. To do this, we suppose that $v(x) \equiv v_\epsilon(x)$ is any positive, sharply peaked function at $x = 0$, satisfying the condition

$$\int_{-\epsilon}^{\epsilon} \mathrm{d}x \, v_\epsilon(x) = 1 \tag{2}$$

ϵ being a small positive parameter. In such a potential, the wavefunction varies rapidly and the dominant terms in (1) are

$$-\mathrm{i}\alpha \frac{\partial \Psi}{\partial x} + \lambda v_\epsilon(x) \simeq 0 \qquad |x| < \epsilon. \tag{3}$$

We note that (3) becomes independent of the self-interaction, so essentially the definition of the point interaction potential is the same as in the linear Dirac equation. As pointed out in [12] and [13], equation (3) admits an iterative Neumann solution of the form

$$\Psi(x) = \hat{P} \exp\left(-\mathrm{i}\alpha\lambda \int_{x_0}^{x} \mathrm{d}y \, v_\epsilon(y)\right) \Psi(x_0) \tag{4}$$

$\hat{P}$ being a Dyson-type ordering operator. Taking the limits $x \to \epsilon$, $x_0 \to -\epsilon$ and $\epsilon \to 0^+$ along with (2), the following boundary condition is reached:

$$\Psi(0^+) = \exp(-\mathrm{i}\alpha\lambda)\Psi(0^-). \tag{5}$$

We should emphasize that the appropriate boundary condition at $x = 0$ becomes independent of how the δ-function limit is taken, so we have overcome the above-mentioned ambiguities. Also, due to the periodicity in λ with period 2π of this boundary condition, we may restrict ourselves to the range $-\pi < \lambda \leqslant \pi$.

In order to find the solutions of (1) one must specify a particular representation for the Dirac matrices. We set $\alpha = \sigma_y$ and $\beta = \sigma_z$, the σ being the Pauli matrices. Bound states are solutions of the form

$$\Psi(x, t) = \exp(-\mathrm{i}\omega t)\psi_\omega(x) \tag{6}$$

with ω real and $\psi_\omega(x)$ vanishing in a suitable way as $|x| \to \infty$. It is convenient to express the upper and lower components of $\psi_\omega(x)$ by means of two real functions $\eta(x)$ and $\vartheta(x)$,

$$\psi_\omega(x) = \eta(x) \begin{pmatrix} \cos\vartheta(x) \\ \sin\vartheta(x) \end{pmatrix}. \tag{7}$$

In terms of these two functions the boundary conditions now read

$$\vartheta(0^+) = \vartheta(0^-) + \lambda \tag{8}$$

$$\eta(0^+) = \eta(0^-) \tag{9}$$

$$\lim_{|x|\to\infty} \eta(x) = 0. \tag{10}$$

It is worth mentioning that $\vartheta(x)$—the relative phase between the two components of ψ_ω—jumps at $x = 0$, whereas $\eta^2(x)$—the charge density—remains continuous at that point.

From (1) and (7) we obtain

$$\mathrm{d}\vartheta/\mathrm{d}x = m\cos 2\vartheta - \omega + \lambda v(x) - 2g\eta^2 \tag{11}$$

$$\mathrm{d}\eta/\mathrm{d}x = m\eta \sin 2\vartheta. \tag{12}$$

We observe that the boundary conditions (8) and (9) may also be obtained by integrating (11) and (12) around $x = 0$, respectively. This ensures that the boundary conditions we have found are consistent and well defined.

Solutions of (11) and (12) satisfying the specified boundary conditions read, for $x \neq 0$,

$$\tan\vartheta(x) = -\frac{\beta}{(m+\omega)} \tanh(\beta x - \gamma\,\mathrm{sgn}(x)) \tag{13}$$

$$\eta^2(x) = \frac{\beta^2/g}{\omega + m\cosh(2\beta|x| - 2\gamma)} \tag{14}$$

where $\mathrm{sgn}(x) = \theta(x) - \theta(-x)$. Here $\beta = +(m^2 - \omega^2)^{1/2}$ must be real to obtain localized solutions, and

$$\gamma = \tanh^{-1}\left(\frac{\beta}{(m-\omega)} \tan(\tfrac{1}{2}\lambda)\right). \tag{15}$$

Note that γ becomes positive (negative) for repulsive (attractive) point interaction potentials because we have chosen $-\pi < \lambda \leqslant \pi$.

We impose the normalization condition (conserved charge)

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \psi_\omega^\dagger \psi_\omega = \int_{-\infty}^{\infty} \mathrm{d}x\, \eta^2 = 1 \tag{16}$$

thus leading to

$$\left(\frac{m-\omega}{m+\omega}\right)^{1/2} = \tan\left(\tfrac{1}{2}g - \tfrac{1}{2}\lambda\right) \tag{17}$$

where we have used (15). Equation (17) can be rewritten in a more compact fashion as

$$\omega = m\cos(g - \lambda). \tag{18}$$

Since the left-hand side of (17) is positive, the condition

$$0 \leqslant \sin(g - \lambda) \tag{19}$$

must be satisfied in order to obtain bound-state solutions. In terms of the two coupling constants, g and λ, the parameter γ is simply given by

$$\gamma = \tanh^{-1}\left(\frac{\tan(\frac{1}{2}\lambda)}{\tan(\frac{1}{2}g - \frac{1}{2}\lambda)}\right). \tag{20}$$

Insofar as $|\tanh z| \leqslant 1$, a further existence condition must be also fulfilled:

$$\cos(g - \lambda) \leqslant \cos\lambda. \tag{21}$$

It is interesting to note that the solutions we have found reduce to well-known results for the unperturbed NLDE [14, 15] in the limit $\lambda \to 0$. In such a limiting case, equation (1) presents localized solutions for all values of $g \in (0, \pi)$, representing truly bound states. As we see from (19) and (21), this is not the case whenever the point interaction potential is introduced, since existence conditions appear. To gain a proper understanding of the above results, we must study attractive and repulsive point interaction potentials separately.

Let us start by considering attractive point interaction potentials, for which $-\pi < \lambda < 0$. The linear Dirac equation for such a potential supports a single bound state with energy $E_{\text{bound}} = m\cos\lambda$, as pointed out by Domínguez-Adame and Maciá [13]. The situation is no longer the same in the case of the NLDE. Condition (19) requires that $g + |\lambda|$ must be in the first two quadrants of the unit circle. For these values of the coupling constants condition (21) is always valid, so equation (1) presents localized solutions whenever

$$g + |\lambda| \leqslant \pi. \tag{22}$$

On increasing the *effective* coupling constant $g + |\lambda|$ the eigenvalue ω decreases, as seen from (18). At the *critical* coupling $g + |\lambda| = \pi$ one finds that $\omega = -m$. Stronger couplings cause no confined solutions at all. This situation is very similar to the appearance of supercritical effects in the linear Dirac equation induced by a strong electrostatic-like potential. In such a case, it is well known that the bound states *dives* into the negative-energy sea as the coupling constant exceeds a certain critical value. It is interesting to mention that recently it has been speculated that a new phase or a soliton-like structure in QED is produced when nonlinear effects in electrodynamics become important and supercritical effects may take place [16].

The occurrence of condition (22) is in contrast to the results found in the non-relativistic version of equation (1), that is, the nonlinear Schrödinger equation (NLSE) plus a δ-function potential. This problem has been recently solved by Pushkarov and Atanasov [17], who demonstrated that there is no condition for the existence of localized solutions of the NLSE with an attractive δ-function potential, provided that the self-coupling is also attractive. Notice that (19) reduces to $g + |\lambda| \geqslant 0$ in the limit of weak coupling, which trivially holds, in accordance to the non-relativistic prediction. In addition, since γ is negative, the charge density η^2 has only one maximum, at the point where the interaction potential is located, as occurs for the NLSE [17].

Now we consider repulsive point interaction potentials, so the coupling constant satisfies $0 < \lambda < \pi$ (for $\lambda = \pi$ the point interaction potential becomes transparent to all energies and is immaterial as far as its effects on the wavefunction are concerned [13].) The linear Dirac equation always presents a single bound state with energy $E_{\text{bound}} = -m\cos\lambda$, as shown in [13]. However, this statement does not remain true for the NLDE. Due to (21), now condition (19) must be fulfilled along with

$$g \geqslant 2\lambda. \tag{23}$$

The last inequality is also valid for the NLSE with a repulsive potential (see equation (17) of [17]). For weak coupling, (21) now leads to $g \geqslant \lambda$, which also directly follows from (23). Hence in the case of weak coupling there exists only one existence condition given by (23), in perfect agreement to that found in the NLSE [17]. In the case of the NLDE with a repulsive point interaction potential γ is positive and η^2 has two maxima situated symmetrically around the point interaction potential, in contrast to the attractive case. The location of the two maxima are given by $x_{\max} = \pm\gamma/m \sin(g - \lambda)$.

Having discussed the existence conditions of localized solutions of the NLDE under the influence of point interaction potentials, we should study the validity of our initial approximations. As mentioned above, we replaced the actual short-ranged potential by a point interaction potential. It is clear that this replacement requires the range R of the external potential to be much smaller than the size of the soliton-type solution. In order to get an estimation of this size we calculate

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, x^2 \psi_\omega^\dagger \psi_\omega = \int_{-\infty}^{\infty} \mathrm{d}x\, x^2 \eta^2. \tag{24}$$

The second moment of the charge density is to be computed numerically from (14) for different values of the coupling constants. Nervertheless, it is possible to find analytical lower bounds, following a similar procedure to that given by Stubbe [18] for the unperturbed NLDE. Multiplying (12) by $x\eta$ after integration yields

$$-2m \int_{-\infty}^{\infty} \mathrm{d}x\, x\eta^2 \sin 2\vartheta = 1 \tag{25}$$

where we have used the normalization condition (16). Then by the Cauchy–Schwartz inequality

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^2\eta^2 \sin^2\vartheta \geqslant \frac{\left(\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x\, x\eta^2 \sin 2\vartheta\right)^2}{\int_{-\infty}^{\infty} \mathrm{d}x\, \eta^2 \cos^2\vartheta} \tag{26}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^2\eta^2 \cos^2\vartheta \geqslant \frac{\left(\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x\, x\eta^2 \sin 2\vartheta\right)^2}{\int_{-\infty}^{\infty} \mathrm{d}x\, \eta^2 \sin^2\vartheta}. \tag{27}$$

On adding (26) and (27) and using (25) we obtain

$$\begin{aligned} \langle x^2 \rangle &\geqslant \frac{1}{16m^2}\left(\int_{-\infty}^{\infty} \mathrm{d}x\, \eta^2 \sin^2\vartheta \int_{-\infty}^{\infty} \mathrm{d}x\, \eta^2 \cos^2\vartheta\right)^{-1} \\ &= \frac{1}{4m^2}\left[1 - \left(\int_{-\infty}^{\infty} \mathrm{d}x\, \eta^2 \cos 2\vartheta\right)^2\right]^{-1}. \end{aligned} \tag{28}$$

The integral appearing in (28) may be calculated analytically. Thus the condition $\langle x^2 \rangle \gg R^2$ holds provided that

$$\left(\frac{1}{2mR}\right)^2 \gg 1 - \left(\frac{4}{g}\right)^2 \sin^2\left(\frac{g}{4}\right) \cos^2\left(\frac{g}{4} - \frac{\lambda}{8}\right). \tag{29}$$

In summary, we have found analytical solutions of the NLDE with vector self-coupling perturbed by an electrostatic-like point interaction potential. In contrast to what occurs in the unperturbed NLDE, we have demonstrated that conditions for the existence of localized solutions arise. These conditions reduce, in the weak coupling limit, to those found in dealing with the analogous problem for the NLSE. We believe that this work is a first step in order to obtain a complete understanding of the interaction between Dirac solitons and point-like potentials. Unlike other well-known nonlinear wave equations (nonlinear Schrödinger and nonlinear Klein–Gordon models in almost all their versions), the propagation of Dirac solitons in disordered media has been ignored (for a review of the *state of the art* of research on nonlinear wave propagation in disordered systems, see [19]). Frequently, the disorder is modelled by a lattice of delta-like impurities with random positions or strengths. Hence it would be interesting to examine the solutions of the NLDE under the influence of a random array of point interaction potentials. This regard could provide a generalization of certain one-dimensional nuclear models (see [20] and references therein), in which quarks are assumed to obey the linear Dirac equation for an array of δ-function potentials, in order to explore nonlinear effects in the binding energy of the nucleus.

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