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Scattering states of relativistic point interaction potentials

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Abstract. Scattering of positive and negative energy Dirac particles moving under the action of vector and scalar point interaction potentials faced to an impenetrable wall has been discussed. The occurrence of a well defined resonance pattern is only found for scalar couplings.

1. Introduction

In a recent paper (Domínguez-Adame and Maciá 1989a), bound states of the one-dimensional Dirac equation for vector plus scalar point interaction potentials have been obtained. The term point interaction potential refers to any arbitrary sharply peaked potential approaching the delta-function limit (zero width and constant area). Such potentials are often used in some physical problems—solid state physics (Domínguez-Adame 1989) or nuclear physics (Domínguez-Adame and Maciá 1989b)—to approximate more complex short-ranged potentials. Vector potential means that the potential is multiplied by the same Dirac matrix as the particle energy, while a scalar potential may be regarded as a position-dependent mass. Therefore, vector potentials couple the charge whereas scalar potentials couple the mass. Hence a short-ranged scalar potential could be originated from the exchange of massive scalar mesons between particles, in the same way as a vector field arises from the exchange of vector bosons.

The aim of this paper is to study the scattering of Dirac particles by point interaction potentials in one-dimension. We discuss virtual states and resonances for both vector and scalar potentials. Using an S-matrix formalism, bound states are also obtained, in agreement with previous works (Domínguez-Adame and Maciá 1989a). We restrict the motion of particles to positive values of the coordinate, so the potential we consider is formed by a point interaction potential in front of an impenetrable wall. Hence the solutions we find may be used as a guide to get some insight into three-dimensional relativistic scattering problems with contact operators (Domínguez-Adame 1990).

2. Relativistic scattering in one dimension

The potential we consider varies in the x direction (say) so the up and down spin states will be degenerate. Hence the wavefunction of the Dirac particle is expressed by just two independent components (Glasser 1983). The Dirac equation in one dimension can be written as ($\hbar = c = 1$)

$$[\alpha p + \beta m + U(x) - E]\psi(x) = 0 \tag{1}$$

where α and β are 2×2 Hermitian traceless matrices with square unit and $\alpha\beta+\beta\alpha=0$ (McKellar and Stephenson 1987). In the standard representation $\alpha=\sigma_x$ and $\beta=\sigma_z$, σ's being the Pauli matrices.

The mixed potential is chosen of the form

$$U(x)=\begin{cases}(g_v+\sigma_z g_s)v(x-a) & x>0\\ \infty & x<0\end{cases} \tag{2}$$

where $v(x)$ is any arbitrary function approaching the δ-function limit, g_v and g_s are the vector and scalar coupling constants, respectively, and $a>0$ is a real parameter.

The point interaction potential may be considered just like boundary conditions on the wavefunction at the point $x=a$. Whereas one could integrate the Schrödinger equation around that point to obtain the required boundary condition, this procedure does not hold for the Dirac equation at all. The reason comes from the fact that the Dirac wavefunction becomes discontinuous for such a potential (in the non-relativistic case, however, only the first derivative of the wavefunction becomes discontinuous (Flügge 1974)), and the integral of $\delta(x)\psi(x)$ is not well defined in strict distribution-theory sense (McKellar and Stephenson 1987). To surmount this difficulty the Dirac equation (1) is written as $\mathrm{d}\psi/\mathrm{d}x=\tilde{G}(x)\psi(x)$, where $\tilde{G}(x)=-\mathrm{i}\sigma_x(\sigma_z m+U(x)-E)$ is a 2×2 matrix. In so doing, the Dirac equation admits a Neumann series solution (by analogy with the time-dependent Schrödinger equation) which only requires the integration of $U(x)$ (approaching the δ-function) between the limits $a-0$ and $a+0$ (this procedure is fully explained by McKellar and Stephenson (1987) and Domínguez-Adame and Maciá (1989a) so we shall omit here any detail). The obtained boundary conditions become independent of the exact form of $v(x)$ and are written in the standard representation as follows

$$\psi(a+0)=\cos(g_v^2-g_s^2)^{1/2}\begin{pmatrix}1 & -\mathrm{i}\alpha_-\\ -\mathrm{i}\alpha_+ & 1\end{pmatrix}\psi(a-0) \tag{3}$$

where $\alpha_\pm=(g_v\pm g_s)\tan(g_v^2-g_s^2)^{1/2}/(g_v^2-g_s^2)^{1/2}$ are always real numbers. Note that the boundary condition (3) becomes periodic for pure vector potentials ($g_s=0$) since $\alpha_\pm=\tan g_v$, while this periodicity is absent for pure scalar potentials ($g_v=0$), because $\alpha_\pm=\pm\tanh g_s$ in that case.

In order to obtain a complete solution of the problem, boundary conditions at the impenetrable wall should also be specified. We shall find again some differences with the non-relativistic case. For relativistic particles the current $\psi^\dagger\sigma_x\psi=2\,\mathrm{Re}(\psi_u^*\psi_l)$ must vanish at $x=0$, since there is no particle for $x<0$ (ψ_u and ψ_l denote here the upper and the lower components of the wavefunction). This can be achieved if ψ_u or ψ_l vanish at $x=0$, although both components cannot simultaneously equal zero, as seen from (1). To get the correct non-relativistic limiting behaviour, i.e. the larger component going down to zero at the wall, we set $\psi_u(0)=0$ for particle solutions and $\psi_l(0)=0$ for antiparticle ones. Hence the wavefunction becomes

$$\psi(x)=\begin{cases}A_+\begin{pmatrix}\sin px\\ \dfrac{-\mathrm{i}p}{E+m}\cos px\end{pmatrix} & 0<x<a\\[2ex] A_+\begin{pmatrix}\sin(px+\delta_+)\\ \dfrac{-\mathrm{i}p}{E+m}\cos(px+\delta_+)\end{pmatrix} & x>a\end{cases} \tag{4a}$$

for $E=+(p^2+m^2)^{1/2}>0$ and

$$\psi(x)=\begin{cases} A_-\begin{pmatrix} \dfrac{-\mathrm{i}p}{E-m}\cos px \\ \sin px \end{pmatrix} & 0<x<a \\ A_-\begin{pmatrix} \dfrac{-\mathrm{i}p}{E-m}\cos(px+\delta_-) \\ \sin(px+\delta_-) \end{pmatrix} & x>a \end{cases} \tag{4b}$$

for $E=-(p^2+m^2)^{1/2}<0$. $A_\pm$ are the amplitudes and $\delta_\pm$ are the phaseshifts, defined in a way exactly analogous to the non-relativistic case. Hereafter the upper and the lower signs refer to positive and negative energies, respectively.

After using the boundary condition at $x=a$, we have for the amplitudes

$$|A_\pm(p)|^{-2}=\cos^2(g_v^2-g_s^2)^{1/2}\left[1+\left(\frac{E\mp m}{E\pm m}\right)\alpha_\mp^2\cos^2 pa+\left(\frac{E\pm m}{E\mp m}\right)\alpha_\pm^2\sin^2 pa\right.$$
$$\left.+\left(\frac{E\pm m}{p}\alpha_\pm-\frac{p}{E\pm m}\alpha_\mp\right)\sin 2pa\right] \tag{5}$$

and for the phaseshifts

$$\tan(pa+\delta_\pm)=\tan pa-\left(\frac{p}{E\pm m}\right)\alpha_\mp\left[1+\left(\frac{E\pm m}{p}\right)\alpha_\pm\tan pa\right]^{-1}. \tag{6}$$

For weak coupling at low energies, both expressions (5) and (6) reduce to the obtained results by solving directly the Schrödinger equation for a δ-function potential in front of an impenetrable wall (Flügge 1974). For strong coupling, however, a relativistic wave equation is indeed required, even at low energies.

Bound states of the potential can be computed directly from the S matrix, as occurs in the non-relativistic treatment. For potentials vanishing beyond some finite distance, the poles of the S matrix in the upper half p plane lie along the imaginary axis, and these poles correspond to bound states of the system (Berestetskii *et al* 1971). The S matrix is related to the phaseshift by $S=S_\pm=\exp(2\mathrm{i}\delta_\pm)=(1+\mathrm{i}\tan\delta_\pm)/(1-\mathrm{i}\tan\delta_\pm)$. From (6) we obtain $S_\pm(p)=\phi_\pm(-p)/\phi_\pm(p)$, where the Jost functions

$$\phi_\pm(p)=1+\left(\frac{E\pm m}{p}\right)\mathrm{e}^{\mathrm{i}pa}\sin pa\,\alpha_\pm+\mathrm{i}\left(\frac{E\mp m}{p}\right)\mathrm{e}^{\mathrm{i}pa}\cos pa\,\alpha_\mp \tag{7}$$

satisfy the usual condition $\phi_\pm(-p^*)=\phi_\pm^*(p)$. We replace p by $\mathrm{i}q$, where $q=+(m^2-E^2)^{1/2}$ is real for bound states. Therefore, the condition for bound states is simply written as $\phi_\pm(\mathrm{i}q)=0$, so the energy levels can be computed from the following transcendental equation

$$-2q=(E\pm m)(1-\mathrm{e}^{-2qa})\alpha_\pm+(E\mp m)(1+\mathrm{e}^{-2qa})\alpha_\mp. \tag{8}$$

Neglecting the effects of the wall on the energy levels (i.e. taking the limit $a\to\infty$ so the particle moves in the whole space), the bound states of a single point interaction potential are obtained, in agreement with previous results (Domínguez-Adame and Maciá 1989a). It is not necessary to solve the trascendental equation (8) to obtain the exact number of bound states supported by the potential, since the Levinson theorem is still valid in the relativistic framework (Ma and Ni 1985, Arshansky and Horwitz 1989 and Klaus 1990). The relativistic generalization of the Levinson theorem, which become more simplified in one-space dimension due to the absence of spin effects or centrifugal barrier, provides a link between the discrete and the continuum parts of the energy spectrum. The theorem requires the calculation of the phaseshift as E approaches $\pm m$. For point interaction potentials, a bound state of positive (negative) energy appears if $\delta_+(0)=\pi$ ($\delta_-(0)=\pi$), while the vanishing of $\delta_+(0)$ ($\delta_-(0)$) indicates the non-existence of bound states of positive (negative) energy. One half-bound state (positive or negative) appears whenever the phaseshift at zero momentum ($\delta_+(0)$ or $\delta_-(0)$ respectively) reaches the value $\pi/2$. Such states are not in fact bound states because the wavefunction is not square-integrable, but they are characterized by the occurrence of an infinite scattering amplitude at zero momentum.

3. Results and discussion

In order to avoid the profusion of free parameters we set $a=m^{-1}$ in our numerical results, i.e. a equals the Compton particle wavelength. Therefore, the remainder parameters are the potential strengths g_v and g_s. Due to the different behaviour of vector and scalar potentials, we shall discuss both cases separately.

3.1. Scalar potential

General features of the scattering states for particles with positive and negative energies, moving under the action of a scalar point interaction potential in front of an impenetrable wall, are shown in figures 1–4.

The amplitude $|A_\pm|^2$ given by (5) is plotted in figure 1. The occurrence of resonance peaks for certain potential strengths is clearly seen in this figure. The resonance pattern depends on the sign of the particle energy. The position of the resonance peaks do not show strong dependence with the potential strength g_s, in agreement with the fact that such positions are related to the energy levels of a particle between two impenetrable walls separated by a distance a. The former assertion could be explained if we consider the asymptotic expression of $|A_+(p)|^2$ for large p values, which is found to be

$$|A_\pm(p)|^2 \sim (1+2\sinh^2 g_s \pm \sin 2pa \sinh 2g_s)^{-1} \qquad p\to\infty. \tag{9}$$

Hence, imposing the extreme condition $\mathrm{d}|A_+|^2/\mathrm{d}p=0$, positions of the peaks are given by $p_n=(2n+1)\pi/4a$, where n is a positive integer, and its allowed values are summarized in the following table

g_s \ E	Positive	Negative
Positive	$n=1,3,\ldots$	$n=2,4,\ldots$
Negative	$n=2,4,\ldots$	$n=1,3,\ldots$

No resonances are associated to the case $n=0$. Note that, although positions described by the condition $p_n=(2n+1)\pi/4a$ has been obtained in the limit of large momenta, figure 1 shows that there exists a good agreement even for low momenta. The extreme

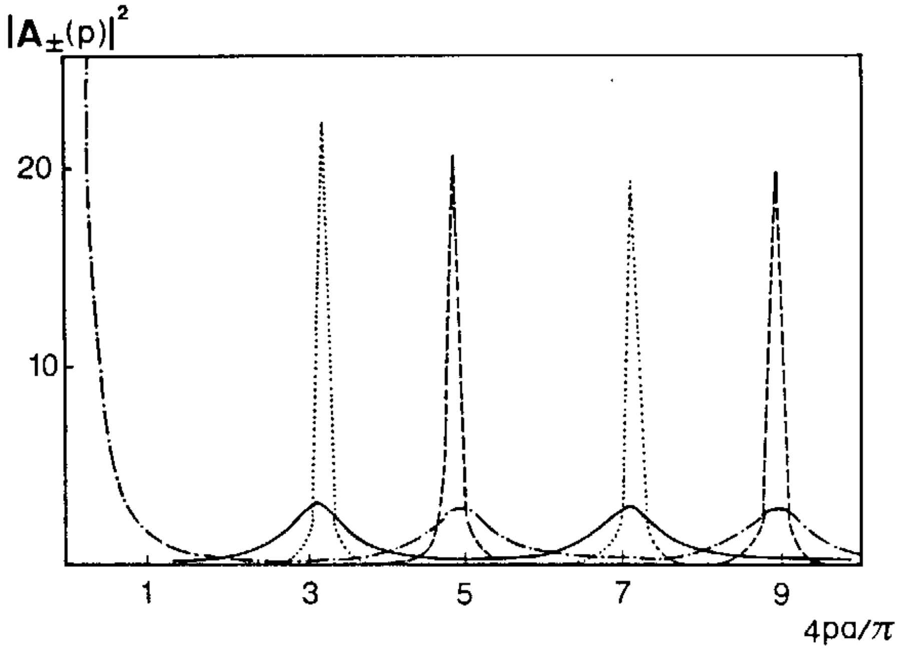

Figure 1. The square modulus of the amplitude for a particle of positive energy subjected to a scalar point interaction potential in front of an impenetrable wall for $g_s = +0.549\ldots$ (full curve), $-0.549\ldots$ (chain curve), 1.5 (dotted curve) and -1.5 (broken curve). Scattering amplitude corresponding to a particle of negative energy is obtained replacing g_s by $-g_s$ in the figure.

condition $\mathrm{d}|A_{\pm}|^2/\mathrm{d}p = 0$ applied directly in (5) leads to the following transcendental equation

$$\tanh g_s = \frac{pa \sin pa - 2p^2a^2(1+p^2a^2)\cos 2pa}{(1+p^2a^2)(2pa \sin 2pa + \cos 2pa) - 2(1+p^2a^2)^{1/2}} \equiv f(p) \tag{10}$$

from whose solutions the position of resonance peaks can be computed. A graphical method for solving (10) is depicted in figure 2 (we consider only positive energies; analogous comments could be stated for negative energies). As seen in this figure, the exact solutions of (10) approach those given by $p_n = (2n+1)\pi/4a$ from the right (left) for repulsive (attractive) potentials as the limit $|g_s| \to 0$ is taken.

The zero-momentum scattering amplitude is given by

$$|A_{\pm}(0)|^2 = (\cosh g_s \pm 2 \sinh g_s)^{-2}$$

which rapidly increases as the potential strength g_v approaches the values $\pm g_s^* \equiv \pm\log\sqrt{3}$. Such a phenomenon is related to the binding properties of the potential, and will be discussed later.

Now let us comment the scattering phaseshift given by (6). From figure 3 is clear that resonances becomes sharper as the strength of the potential increases but, unlike the non-relativistic case (Van Sinclen 1988), the sign of the phaseshift does not appear to be correlated to the attractive or repulsive character of the potential. In the vanishing particle momentum limit we have

$$\lim_{p\to 0} \delta_{\pm}(p) = \begin{cases} 0 & g_s > -g_s^* \\ \pi/2 & g_s = -g_s^* \\ \pi & g_s < -g_s^*. \end{cases} \tag{11}$$

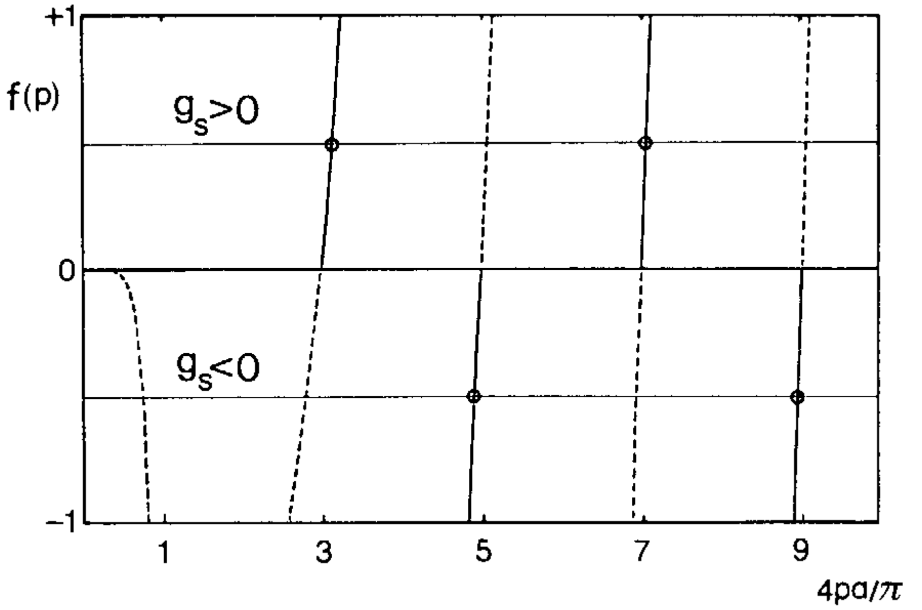

Figure 2. Graphical method to obtain the position of the amplitude resonant peaks corresponding to positive energy particles under scalar coupling. Open circles denote the position of resonance peaks for positive (upper half-plane) and negative (lower half-plane) values of the scalar strength g_s.

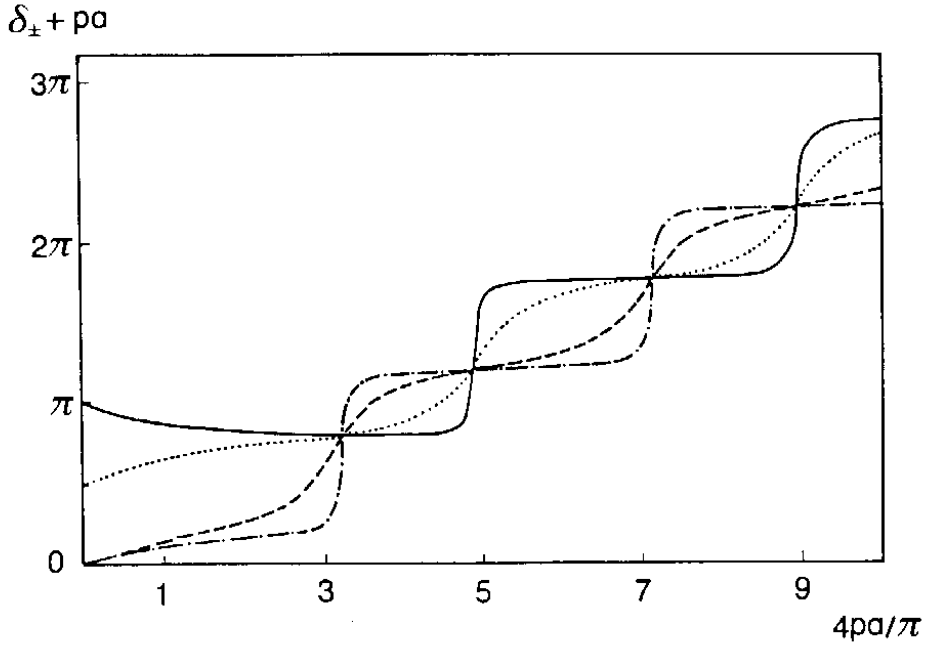

Figure 3. Scattering phaseshift plotted as a function of pa for particles of both positive and negative energies subjected to a scalar coupling for $g_s = -0.549\ldots$ (dotted curve), $+0.549\ldots$ (broken curve), -1.5 (full curve) and 1.5 (chain curve).

The limit given in (11) partially agrees with the non-relativistic result obtained by Van Sinclen. The main difference is the exact value of g_s^*; in our results we have obtained $g_s^* = 0.549\ldots$ while Van Siclen has deduced a value of 0.5 for non-relativistic particles. According to the Levinson theorem (Ma and Ni 1985), two half-bound states (positive and negative) occur as g_s equals the critical value $-g_s^*$. Starting from a potential strength larger than $-g_s^*$, we note that only scattering states may occur; by decreasing g_s until the critical value $-g_s^*$ is reached; two half-bound states appear. By further decreasing of g_s just below $-g_s^*$ one bound state of either energy sign occurs, and the phaseshift at $p = 0$ jumps to π. Thus the phaseshift evolution is not continuous. Instead it shows a clear cut-off between binding and non-binding potentials. As we pointed out above, just for the critical coupling $g_s = -g_s^*$ the value $\delta_\pm(0) = \pi/2$ is reached. However, this jumping behaviour is a typical feature describing the particle phaseshift evolution passing through a resonant state. Thus, from the above considerations, we are led to the conclusion that a zero-momentum particle interacting with a potential of strength $g_s = -g_s^*$ undergoes a resonance rather than a bound or virtual state. Note that the amplitude becomes infinite at $p = 0$ as $g_s = -g_s^*$ only for positive energies. On the other hand one finds an infinite amplitude at $p = 0$ as $g_s = +g_s^*$ for negative energies as well, but in this case $\delta_\pm(0) = 0$, so it should be considered as a virtual rather than a resonance state.

To get further insight, we study the S-matrix poles given in equation (8). The bound state energy must satisfy the relationship

$$\tanh g_s = \frac{qa}{1-(1-q^2a^2)^{1/2}\exp(-2qa)} \tag{12}$$

where $E = \pm(m^2 - q^2)^{1/2}$. Since the right-hand side of (12) remains always negative, only attractive potentials could support bound states at first. Taking the limit $q \to 0$ in (12) we obtain $g_s(q = 0) = -\tanh^{-1}(1/2) = -\log\sqrt{3} = -g_s^*$. Therefore, only attractive potentials with g_s less than $-g_s^*$ present binding of particles, no matter the sign of the energy (i.e. bound states do appear in pairs (Coutinho and Nogami 1987) since (12) does not depend on the sign of the energy). This treatment completely agrees with our previous discussion about the phaseshift (11). Binding properties of the system as a function of the scalar potential strength are depicted in figure 4.

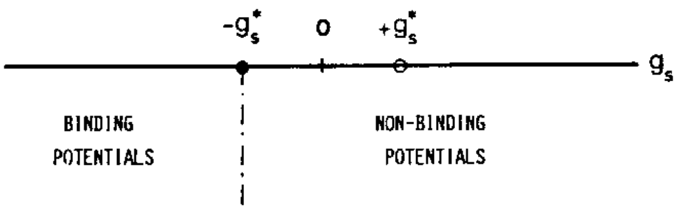

Figure 4. Binding properties of the system as a function of the scalar potential strength. Open and full circles denote the occurrence of a virtual state and of a zero-momentum resonance, respectively.

3.2. *Vector potential*

Figure 5 shows the scattering amplitude for positive and negative energy states. A comparison between this figure and figure 1 clearly demonstrates that resonant peaks

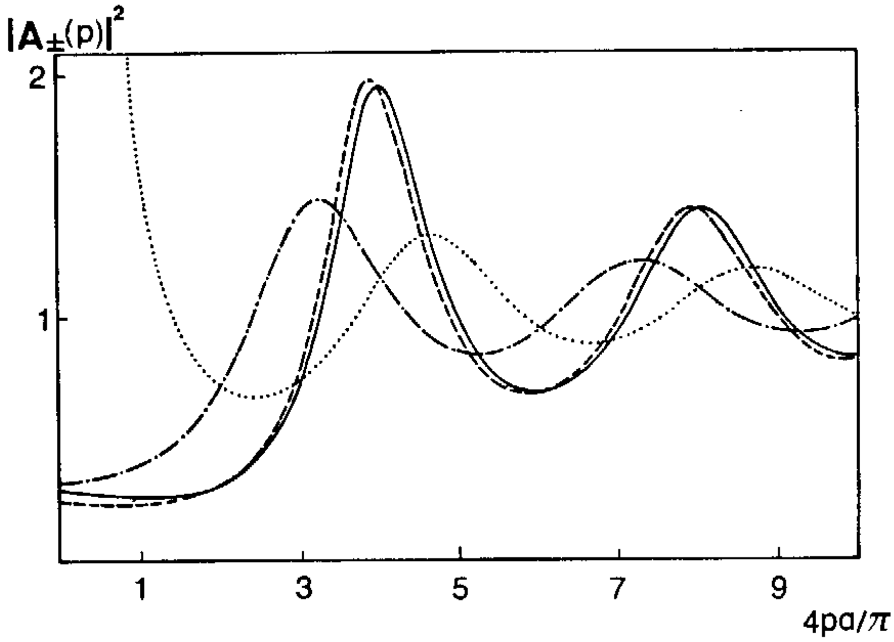

Figure 5. The square modulus of the amplitude for a particle of positive energy subjected to a vector point interaction potential in front of an impenetrable wall for $g_v = 0.464\ldots$ (dotted curve), $-0.464\ldots$ (chain curve), -1.5 (broken curve) and 1.5 (full curve). Scattering amplitude for a particle of negative energy is obtained replacing g_v by $-g_v$ in the figure.

are lower and broader for vector couplings than for scalar ones. In fact, for a given g_v, resonant peaks slowly decrease and approach the limiting value $|A_+|^2 \rightarrow 1$ for large momenta. As a consequence, high energy particles in a vector point interaction potential faced to an impenetrable wall do not show resonances at all.

The extreme condition applied to the amplitude (5) as $g_v \neq 0$ and $g_s = 0$ leads to the following conditions

$$\sin g_v = 0 \tag{13a}$$

$$\tan g_v = \frac{p^2a^2(1+p^2a^2)^{1/2}\sin pa - 2p^2a^2(1+p^2a^2)\cos 2pa}{(1+p^2a^2)(2pa\sin pa + \cos 2pa) - 2(1+p^2a^2)^{1/2}}. \tag{13b}$$

The set of solutions related to (13a) corresponds to $g_v = n\pi$, n being an integer. For these values of g_v the potential becomes absolutely transparent to all energies, as can be deduced from a detailed analysis of the transmission coefficient (Domínguez-Adame and Maciá 1989a). Solutions of (13b) are associated with both maxima and minima amplitude positions. Such solutions are more difficult to obtain than for the pure scalar potential case, but once again peak positions do not strongly depend on g_v, as seen in figure 5.

The limit of the amplitude at zero-momentum is readily found to be $|A_\pm(0)|^2 = (\cos g_v \pm 2\sin g_v)^{-2}$, resembling the scalar potential case. Nevertheless, now we have an infinite set of coupling constants for which the amplitude at the origin becomes infinite. These special values of the coupling constant are $g_v = \mp g_v^*$, where $g_v^* = \tan^{-1}(1/2) + n\pi = +0.464\ldots + n\pi$, n being an integer. Due to this periodicity we consider the case $n = 0$; the obtained conclusions also hold for $n \neq 0$.

Condition for the existence of bound states may be written from (8) as

$$\tan g_v = \pm \frac{qa}{\exp(-2qa) - (1 - q^2a^2)^{1/2}} \tag{14}$$

where we can restrict ourselves to the range $-\pi \leqslant g_v \leqslant \pi$ because of the periodicity of the boundary condition on g_v. There exist real, positive q values corresponding to bound states if $g_v^* < |g_v| < \pi$ whereas the potential possesses only scattering states if $0 < |g_v| < g_v^*$. This result agrees with the limiting behaviour of the phaseshift at low particle momenta

$$\lim_{p\to 0} \delta_+(p) = \begin{cases} 0 & -g_v^* < g_v < \pi \\ \pi/2 & g_v = -g_v^* \\ \pi & -\pi < g_v < -g_v^* \end{cases}$$

and

$$\lim_{p\to 0} \delta_-(p) = \begin{cases} 0 & -\pi < g_v < g_v^* \\ \pi/2 & g_v = g_v^* \\ \pi & g_v^* < g_v < \pi. \end{cases} \tag{15}$$

Apart from the jump of the phaseshift at $p = 0$ as g_v approaches $\mp g_v^*$, one can observe in figure 6 that $\delta_\pm(p)$ increases only smoothly. From the results just presented some conclusions can be drawn. First, the presence of resonant states (if any) is of minor importance for vector-type point interaction potentials as compared with scalar-type ones. Second, the amplitude and the phase plots exhibit great resemblance for both signs of the particle energy. Finally, only potentials satisfying $|g_v| > g_v^*$ can support

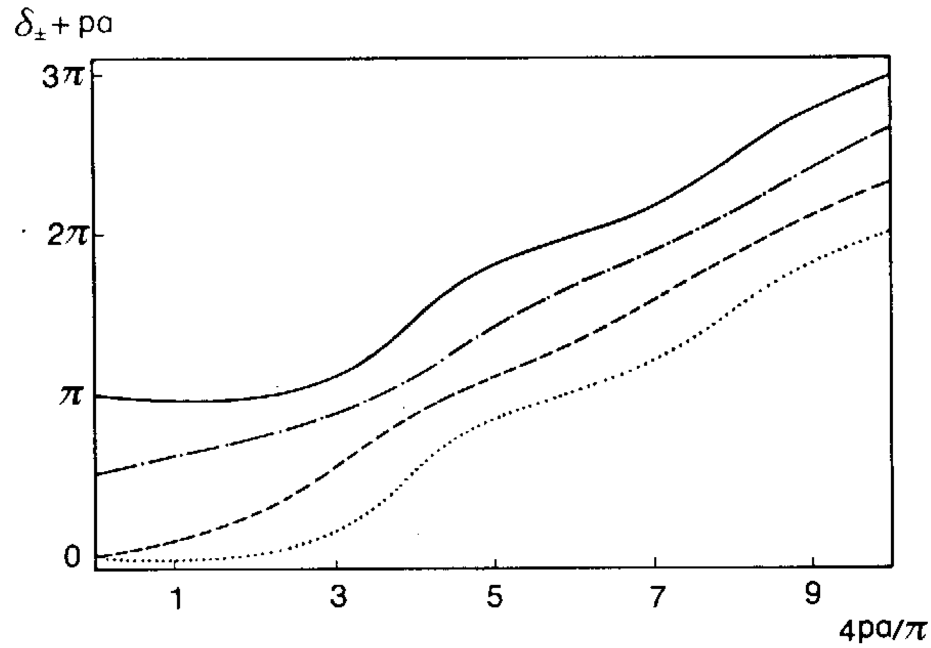

Figure 6. Scattering phaseshift plotted as a function of pa for positive energy particles subjected to a vector coupling for $g_v = 0.464\ldots$ (chain curve), $-0.464\ldots$ (broken curve), -1.5 (full curve) and 1.5 (dotted curve). Phaseshift corresponding to negative energy particles can be obtained replacing g_v by $-g_v$ in this figure.

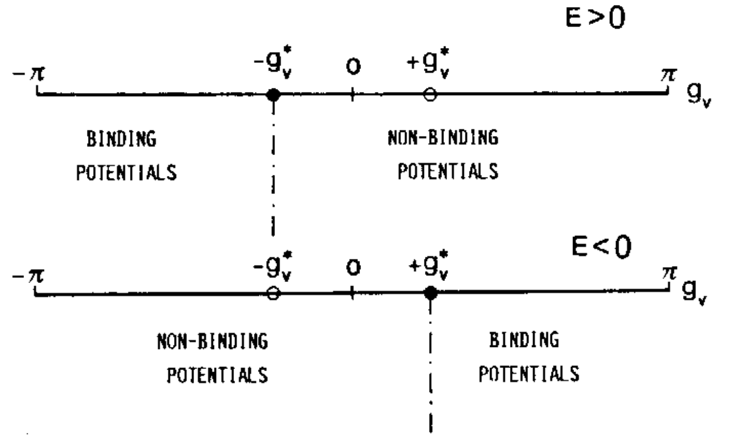

Figure 7. Binding properties of the system as a function of the vector potential strength for both signs of the particle energy. Open and full circles denote the occurrence of a virtual state and of a zero-momentum resonance, respectively.

bound states (see figure 7) while half-bound states occur if $|g_v| = g_v^*$. The 'forbidden' region of potential strength ranging from $-g_v^*$ to $+g_v^*$ disappears as the distance between the point interaction potential and the wall becomes very large. Then, the bound state spectra of an isolated vector point interaction potential is obtained (Domínguez-Adame and Maciá 1989a).

4. Summary

The critical values of the potential strength separating binding and non-binding potentials have been found to be $-0.549\ldots$ for scalar couplings and $\mp 0.464\ldots + n\pi$ for vector ones in our relativistic treatment. These values are slightly different from that obtained in the non-relativistic framework which is calculated to be -0.5. The main difference between vector and scalar couplings in relation to the scattering properties concerns the phaseshift behaviour at resonances. While $\delta_1(pa) + pa$ shows a clear step-like evolution as pa increases for the scalar point interaction potentials, a smooth behaviour is found for vector ones. Also, in the former potential resonance peaks were narrower and higher, thus revealing the occurrence of well-defined resonance states only for scalar couplings.

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