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Relativistic Particles in Orthogonal Electric and Magnetic Fields with Confining Scalar Potentials (*).

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Summary. — The motion of Klein-Gordon and Dirac particles in constant crossed electric and magnetic fields in addition to a confining scalar potential is studied in momentum space. The existence of bound states, as the electrostatic coupling is smaller than the magnetic and scalar couplings, is found.

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1. - Introduction.

Relativistic wave equations with electrostatic and scalarlike linear potentials have been widely used in phenomenological models to investigate the confinement of quarks. Here the term electrostatic means the time component of a Lorentz vector, while a scalar potential is equivalent to a dependence of the rest mass upon position. If the potential is electrostaticlike, there exist no bound states [1], and only tunnelling solutions arise so that such a potential is not in fact confining. This is another example of the famous Klein paradox. On the contrary, scalar linear potentials can bind relativistic particles [2,3], giving rise to confinement. Also for a mixture of these potentials, it is well established that confinement only occurs if the scalar term is stronger than the electrostatic term [4,5]. All these conclusions are valid for $(3+1)$- and for $(1+1)$-dimensional relativistic wave equations, and have been demonstrated by solving the equations in the position space.

In this paper we discuss the confining properties of the Klein-Gordon and Dirac equations for electrostatic and scalar linear potentials varying in one-space dimension, in addition to a constant crossed magnetic field. Besides its purely methodological interest, this configuration allows us to know more exactly how relativistic quarks behave in an external magnetic field, as occurs in some astrophysical problems. Relativistic particles in constant crossed electric and

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magnetic fields (a uniform electric field is a simple realization of an electrostatic linear potential) were previously studied by Lam [6, 7] in the position representation. This author clearly demonstrated that the quantization of energy levels arises if the strength of the magnetic field is larger than the strength of the electric field. The solution found by Lam, however, cannot be applied to study the behaviour of quarks in external magnetic fields since the existence of a scalar linear potential is required to confine quarks.

We also show that the motion of relativistic particles under the action of electrostatic and scalar linear potentials with an orthogonal magnetic field may be successfully described in momentum space. We find analytical solutions for the Klein-Gordon and the Dirac equations, which include as special cases those discussed in position space by Sun and Yuhong [4] (electrostatic and scalar linear interactions) and by Lam [6, 7] (crossed constant electric and magnetic fields).

2. – Klein-Gordon equation.

To get insight into the problem we start with the Klein-Gordon equation for a spinless particle of rest mass m and charge q

$$\{(\bar{\boldsymbol{p}} - q\bar{\boldsymbol{A}})^2 + (m + S)^2 - (E - qV)^2\}\psi = 0\,, \tag{1}$$

where the Lorentz vector potential is $(V, \bar{\boldsymbol{A}})$ and S denotes the Lorentz scalar potential. We suppose that V and S vary only along the x-direction

$$qV = K_V x\,, \tag{2a}$$

$$S = K_S x \tag{2b}$$

and $\bar{\boldsymbol{A}}$ describes a uniform magnetic field $\bar{\boldsymbol{B}} = \nabla \times \bar{\boldsymbol{A}}$ in the z-direction, so we can set the gauge

$$q\boldsymbol{A} = (0, K_B x, 0)\,. \tag{2c}$$

Therefore, the Klein-Gordon equation reduces to

$$\begin{aligned}[p_x^2 + 2(mK_S + EK_V - p_y K_B)x &+ \\ &+ (K_B^2 + K_S^2 - K_V^2)x^2 - (E^2 - m^2 - p_y^2 - p_z^2)]\psi(x) = 0\,.\end{aligned} \tag{3}$$

Since one can replace $x \to i(\mathrm{d}/\mathrm{d}p_x)$, we can take advantage of the form of the potential to cast the problem in momentum space. Thus, eq. (3) reads

$$\begin{aligned}\Bigg[(K_B^2 + K_S^2 - K_V^2)\frac{\mathrm{d}^2}{\mathrm{d}p_x^2} - 2i(mK_S + EK_V - p_y K_B)\frac{\mathrm{d}}{\mathrm{d}p_x} &+ \\ + E^2 - m^2 - p_x^2 - p_y^2 - p_z^2\Bigg]\psi(p_x) &= 0\,.\end{aligned} \tag{4}$$

We have assumed an $\exp[i(p_y y + p_z z)]$ dependence for the wave function since both p_y and p_z are constant of motion.

In the particular case $K_B^2 + K_S^2 - K_V^2 = 0$, eq. (4) reduces to a first-order differential equation, whose solution is readily found to be

$$\psi(p_x) = \psi(0)\exp\left[ip_x \frac{p_x^2 - 3(E^2 - m^2 - p_y^2 - p_z^2)}{6(mK_S + K_V - p_y K_B)}\right]. \tag{5}$$

The particle wave function rapidly oscillates for large momentum, being unnormalizable in the usual fashion (the solution is not included in L^2). Hence there exist no bound states in this case, no matter the sign of the coupling constants. The result may also be easily understood by noting that eq. (3) with $K_B^2 + K_S^2 - K_V^2 = 0$ becomes a Schrödinger-like equation for a uniform electric field, which is known to present no bound states.

We now seek for the solutions of the Klein-Gordon equation in the general case $K_B^2 + K_S^2 - K_V^2 \neq 0$, which is then written as

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}u^2} - 2i\xi\frac{\mathrm{d}}{\mathrm{d}u} - u^2 + \eta\right\}\psi(u) = 0\,, \tag{6}$$

where we have defined the following dimensionless quantities:

$$\begin{cases} u \equiv p_x(K_B^2 + K_S^2 - K_V^2)^{-1/4}\,, \\ \xi \equiv (mK_S + EK_V - p_y K_B)(K_B^2 + K_S^2 - K_V^2)^{-3/4}\,, \\ \eta \equiv (E^2 - m^2 - p_y^2 - p_z^2)(K_B^2 + K_S^2 - K_V^2)^{-1/2}\,. \end{cases} \tag{7}$$

The variable u^2 and the parameters ξ^2, η may be real ($K_B^2 + K_S^2 > K_V^2$) or purely imaginary ($K_B^2 + K_S^2 < K_V^2$). This fact completely determines the asymptotic behaviour of ψ for large momentum.

Assuming an $\exp[-u^2/2 + i\xi u]$ dependence of the wave function $\psi(u)$, we obtain a confluent hypergeometric equation in the variable u^2 from eq. (4). The solution is

$$\psi(u) = \exp[-u^2/2 + i\xi u]\,[A\,{}_1F_1((1-\eta-\xi^2)/4,\, 1/2;\, u^2) + {} + Bu\,{}_1F_1((3-\eta-\xi^2)/4,\, 3/2;\, u^2)]\,, \tag{8}$$

A, B being two arbitrary constants. To find conditions for the existence of bound states (if any), we require the wave function ψ to be square integrable. Therefore, we must investigate the asymptotic form of ψ at high momenta. Using the expansions of the confluent hypergeometric functions for a large argument [8] one gets

$$\psi(u) \sim \exp[-u^2/2 + i\xi u]\,u^{(1-\eta-\xi^2)/2}\cdot [A/\Gamma((1+\eta+\xi^2)/4) + B/\Gamma((3+\eta+\xi^2)/4)] + \exp[+u^2/2 + i\xi u]\,u^{-(1+\eta+\xi^2)/2}\cdot [A/\Gamma((1-\eta-\xi^2)/4) + B/\Gamma((3-\eta-\xi^2)/4)]\,, \qquad |u| \to \infty\,, \tag{9}$$

where some constant factors have been absorbed in A and B. We now discuss two possible situations.

i) In the case $K_B^2 + K_S^2 > K^2$, the variable u and the parameters η, ξ are real. Therefore, to get square integrable solutions, the second term of (9) must vanish.

This may be accomplished requiring that

$$A = 0 \qquad (3 - \eta - \xi^2)/4 = 0,\ -1,\ -2\ldots \tag{10a}$$

or

$$B = 0 \qquad (1 - \eta - \xi^2)/4 = 0,\ -1,\ -2\ldots \tag{10b}$$

because only for these special values of η and ξ^2 the gamma-function Γ becomes infinite. Thus we are led to the conclusion that there exist bound states if $K_B^2 + K_S^2 > K_V^2$. In fact, the entire spectrum becomes discrete. Bound-state solutions may be written in terms of Hermite polynomials

$$\psi(u) = \exp[-u^2/2 + i\xi u]\,H_{n-1}(u) \tag{11}$$

and the corresponding energy levels are given by

$$(EK_B - p_y K_V)^2 + (EK_S + mK_V)^2 - (mK_B + p_y K_S)^2 - (K_B^2 + K_S^2 - K_V^2)\,p_z^2 = = (2n-1)(K_B^2 + K_S^2 - K_V^2)^{3/2}, \tag{12}$$

n being a positive integer. For pure linear scalar potential ($K_B = K_V = 0$, $K_S \neq 0$) the energy levels appear in pairs

$$E = \pm(p_y^2 + p_z^2 + (2n-1)|K_S|)^{1/2} \tag{13a}$$

and become independent of the particle mass. For the motion of spinless particles in a uniform magnetic field, eq. (12) reduces to ($K_S = K_V = 0$, $K_B \neq 0$)

$$E = \pm(m^2 + p_z^2 + (2n-1)|K_B|)^{1/2} \tag{13b}$$

and also the discrete spectrum becomes symmetric around $E = 0$.

ii) The situation is quite different in the opposite case, *i.e.* $K_B^2 + K_S^2 < K_V^2$. From (7) we note that u^2, ξ^2, η and $i\xi u$ become purely imaginary. Therefore, the gamma-functions appearing in (9) never reach infinite, so the wave function decreases as $\psi(u) \sim u^{-1/2}$ at spatial infinity, being unnormalizable in the usual fashion. We conclude that particles cannot be bounded if the electrostatic potential is strong, as K_V^2 is larger than $K_B^2 + K_S^2$. Only scattering solutions appear and the whole spectrum is continuum. Solutions may be given by means of parabolic cylinder functions

$$\psi(u) = (C_+ D_{-1/2+\eta+\xi^2}(\sqrt{2}\,u) + C_- D_{-1/2+\eta-\xi^2}(\sqrt{-2}\,u))\exp[i\xi u], \tag{14}$$

$C_\pm$ being two constants.

3. – The Dirac equation.

In this section we consider relativistic particles with spin one-half, described by the Dirac equation. The Dirac equation for a Lorentz potential $(V, \boldsymbol{A})$ plus a scalar potential is written as

$$\{\bar{\alpha}\cdot(\bar{\boldsymbol{p}} - q\bar{\boldsymbol{A}}) + \beta(m + s) - (E - qV)\}\,\psi = 0, \tag{15}$$

where we choose the representation

$$\bar{\alpha} = \begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{16}$$

with $\bar{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and I_n stands for the $n \times n$ unit matrix hereafter. To solve the Dirac equation, we use the ansatz

$$\psi = \{\bar{\alpha} \cdot (\bar{\boldsymbol{p}} - q\bar{\boldsymbol{A}}) + \beta(m + S) + (E - qV)\} \chi, \tag{17}$$

where the four-component function χ satisfies the equation

$$\{(\bar{\boldsymbol{p}} - q\bar{\boldsymbol{A}})^2 + (m + S)^2 - (E - qV)^2 - q\bar{\sigma} \cdot \bar{\boldsymbol{B}} I_4 - [\bar{\boldsymbol{p}}, \beta S + qV]\bar{\alpha}\} \chi = 0, \tag{18}$$

$\bar{\boldsymbol{B}} = \nabla \times \bar{\boldsymbol{A}}$ being the magnetic field. Therefore, we find that χ can be calculated by solving a Klein-Gordon equation containing some nondiagonal terms due to the interaction of the spin with the external potentials. For the potentials given in (2), eq. (18) may be written in momentum space as

$$\begin{aligned} &\Bigg\{(K_B^2 + K_S^2 - K_V^2) \frac{\mathrm{d}^2}{\mathrm{d}p_x^2} - 2i(mK_S + EK_V - p_x K_B) \cdot \\ &\cdot \frac{\mathrm{d}}{\mathrm{d}p_x} - p_x^2 - p_y^2 - p_z^2 + E^2 - m^2 + K_B \sigma_z I_4 - i(\beta K_S + K_V)\alpha_x \Bigg\} \chi(p_x) = 0. \end{aligned} \tag{19}$$

The matrix operator $+K_B \sigma_z I_4 - i(\beta K_S + K_V)\alpha_x$ becomes independent of p_x, so the asymptotic behaviour of eq. (19) at large values of p_x is the same as that of the Klein-Gordon equation. Therefore, we conclude that there exist bound states only if $K_B^2 + K_S^2 > K_V^2$, whereas scattering states occur for $K_B^2 + K_S^2 \leqslant K_V^2$. However, we will explicitly solve the Dirac equation in the general case $K_B^2 + K_S^2 - K_V^2 \neq 0$ to study spin effects and to find analytical solutions in the momentum representation. We introduce the dimensionless quantities defined in (7), so eq. (19) reads

$$\left\{ \frac{\mathrm{d}^2}{\mathrm{d}u^2} - 2i\xi \frac{\mathrm{d}}{\mathrm{d}u} - u^2 + \eta + \tilde{M} \right\} \chi(u) = 0, \tag{20}$$

where the matrix $\tilde{M}$ is given by

$$\tilde{M} = (K_B^2 + K_S^2 - K_V^2)^{-1/2} \begin{pmatrix} K_B & 0 & 0 & -i(K_S + K_V) \\ 0 & -K_B & -i(K_S - K_V) & 0 \\ 0 & i(K_S - K_V) & K_B & 0 \\ i(K_S - K_V) & 0 & 0 & -K_B \end{pmatrix}, \tag{21}$$

whose eigenvalues are $\lambda = \pm 1$. Hence solutions of eq. (20) may be constructed as follows:

$$\chi(u) = f_\lambda(u) u_\lambda, \tag{22}$$

where u_λ is an eigenvector of $\tilde{M}$ with eigenvalue λ, and the function $f_\lambda(u)$ satisfies

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}u^2} - 2i\xi\frac{\mathrm{d}}{\mathrm{d}u} - u^2 + \eta + \lambda\right\}f_\lambda(u) = 0 \tag{23}$$

which only differs from eq. (6) in the last term. Therefore, we can readily discuss the following two cases:

i) Weak electrostatic term, $K_S^2 + K_B^2 > K_V^2$. In this case the particle may be confined with an energy given by

$$(EK_B - p_y K_V)^2 + (EK_S + mK_V)^2 - (mK_B + p_y K_S)^2 - (K_B^2 + K_S^2 - K_V^2)p_z^2 = \\ = 2(n + 1/2 - \lambda/2)(K_B^2 + K_S^2 - K_V^2)^{3/2} \tag{24}$$

with $n = 1, 2 \ldots$. Therefore, the energy levels of Dirac particles are given in terms of positive even integers $2n + 1 - \lambda = 2, 4 \ldots$, whereas the energy levels of spinless particles are given by means of positive odd integers $2n - 1 = 1, 3 \ldots$. The auxiliary function $f_\lambda(u)$ is found to be

$$f_\lambda(u) = \exp[-u^2/2 + i\xi u]H_{n-1}(u) \tag{25}$$

and the corresponding Dirac wave function may be obtained with the aid of (17) and (22), which results in a combination of Hermite polynomials.

ii) Strong electrostatic term, $K_V^2 > K_S^2 + K_B^2$. As occurs for spinless particles, Dirac particles cannot be confined in this case, and only scattering solutions exist. The solution of (23) is

$$f_\lambda(u) = (C^\lambda + D_{-1/2+\eta+\lambda+\xi^2}(\sqrt{2}\,u) + C^\lambda - D_{-1/2-\eta-\lambda-\xi^2}(\sqrt{-2}\,u))\exp[i\xi u], \tag{26}$$

where $C_\pm^\lambda$ denotes two arbitrary constants.

4. – Conclusions.

We may conclude that relativistic particles in constant crossed electric and magnetic fields in addition to a linear scalar potential may effectively be confined, whenever the electrostatic coupling is properly limited compared to the scalar and magnetic couplings. In such a case, the spin of the particle only shifts the energy levels in comparison to the energy of spinless particles, but has no effects on the confining properties of the potential. On the other side, we have only found scattering states for strong electrostatic couplings, so particles are not confined. To avoid the leakage of particles (Klein paradox), one must consider additional conditions on the wave function. Thus a «cut-off» $P < \infty$ for large values of the particle momentum leads to the occurrence of bound states, because the wave function becomes square integrable. The energy levels are found through the condition $\psi(P) = 0$. This procedure is equivalent to restrict the motion of particles between two hard walls separated by a distance $\Delta x < \infty$, so electrostatic linear potentials present binding of particles [9]. A similar argument to that given above has recently been suggested to regularize overcritical Coulomb potentials in momentum space [10].

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