ELSEVIER

Physica B 307 (2001) 247–250

PHYSICA B

www.elsevier.com/locate/physb

Eigenvalues of non-Hermitian Fibonacci Hamiltonians

F. Domínguez-Adame*

GISC, Departamento de Física de Materiales, Facultad de Cienicas Fisicas, Universidad Complutense, de Madrid E-28040 Madrid, Spain

Received 26 June 2001

Abstract

We study, using numerical methods, the effects of a non-Hermitian field h on the eigenvalues of a tight-binding Fibonacci system. For vanishing non-Hermitian field, all eigenvalues are real and correspond to critical eigenfunctions. The eigenvalues become complex and eigenfunctions tend to be delocalized for non-zero values of the parameter h. The transition from critical to extended states is monitored through the inverse participation ratio as a function of h. A simple two-band model is introduced to explain the behavior of the eigenvalues on the complex plane. © 2001 Elsevier Science B.V. All rights reserved.

PACS: 73.20.Jc; 73.22.Dj; 74.60.Ge; 61.44.Fw

Keywords: Delocalization; Flux pinning; Incommensurability; Non-Hermitian Hamiltonian; Fibonacci system

1. Introduction

Pinning of vortices by columnar defects in superconductors has been successfully described by non-Hermitian random Hamiltonians [1–3]. Further noteworthy applications of non-Hermitian random Hamiltonians across several areas of physics are collected in Refs. [4–6]. A key ingredient in those models is the competition between the diagonal disorder and the non-Hermitian field. While it is generally accepted that all eigenstates in one-dimensional (1D) disordered Hermitian Hamiltonians are localized and, consequently, the Anderson transition is absent in such systems [7], the introduction of a non-Hermitian field may result in a transition from localized to delocalized states [1,8].

While much work has been devoted to non-Hermitian random Hamiltonians, less attention received other types of ordering. In particular, correlated disorder and aperiodic order are now routinely used to describe physical properties of new materials. As an example of long-range correlated non-Hermitian Hamiltonian, we focus the attention on the archetypal Fibonacci sequence. There exists a large number of works dealing with the energy spectrum and eigenfunctions of Hermitian Fibonacci Hamiltonians (both diagonal and off-diagonal), but the problem of pinning of vortices by long-range correlated columnar defects is still unresolved. In this work we share some light into this and related problems which are described by non-Hermitian aperiodic Hamiltonians and, in particular, by non-Hermitian Fibonacci Hamiltonians.

*Tel.: +34-91-394-44-88; fax: +34-91-394-45-47.
E-mail address: adame@valbuena.fis.ucm.es (F. Domínguez-Adame).

PII: S0921-4526(01)00962-0

2. Model

Here, we consider a 1D non-Hermitian aperiodic Schrödinger equation

$$e^{h}\psi_{n+1} + e^{-h}\psi_{n-1} + \varepsilon_n\psi_n = E\psi_n, \quad (1)$$

with the periodic identification $i + N \equiv i$ of site indices. The diagonal part ε_n is set from a binary Fibonacci sequence with basic energy units ε_A and ε_B. A usual method to construct the Fibonacci sequence is to use an inflation process according to the rule $A \to AB$ and $B \to A$. This procedure yields the following site-energy sequence $\varepsilon_A\varepsilon_B\varepsilon_A\varepsilon_A\varepsilon_B \ldots$. The number of sites is given by $N \equiv F_\ell$, where the Fibonacci numbers satisfy $F_\ell = F_{\ell-1} + F_{\ell-2}$ ($\ell \geqslant 2$) with $F_0 = F_1 = 1$. Hereafter we take $\varepsilon_A = 0$, $\varepsilon_B \geqslant 0$ and $h \geqslant 0$ without loss of generality.

Without non-hermiticity ($h = 0$ and $\varepsilon_n \neq 0$) all eigenvalues are real and the Hamiltonian corresponds to a standard diagonal Fibonacci system [9]. The spectrum is then purely singular continuous, self-similar and highly fragmented, namely the number of subbands increases on increasing the system size and, at the same time, their widths progressively decrease. Consequently, the eigenfunctions are critical [10], independently of the system parameters, so they are not extended nor localized in the standard fashion but present large spatial fluctuations at all length scales.

In the opposite limit ($h \neq 0$ and $\varepsilon_n = 0$), the resulting non-Hermitian Hamiltonian is exactly solvable as it corresponds to an ordered system. Bloch theorem ensures that the eigenvalues are given by $E_k = 2\cosh(ik + h)$ for an infinite system [11]. Thus, eigenvalues become complex and trace out an ellipse on the complex plane. Semiaxes are $2\cosh h$ and $2\sinh h$ on the real and imaginary axis, respectively. The corresponding eigenfunction $\psi_n^{(k)} \sim \exp(\mathrm{i}nk)$ are delocalized, as expected.

3. Numerical results

To study the transition from critical ($h = 0$ and $\varepsilon_n \neq 0$) to delocalized ($h \neq 0$ and $\varepsilon_n = 0$) eigenfunctions, we have numerically diagonalized the non-Hermitian Hamiltonian (1) for different values of the model parameters (h and N). We will fix $\varepsilon_B = 1$ hereafter. For any set of parameters, if E_k is an eigenvalue of the non-Hermitian Hamiltonian (1), then also E_k^* is one because the Hamiltonian is real. Fig. 1 shows that this is indeed the case. The spectrum entirely lies over the real axis for $h = 0$ since the Hamiltonian is Hermitian. The eigenvalues are grouped together forming subbands separated by well-defined subgaps (labeled A, B and C in the figure). In fact, Liu and Riklund established that *diagonal* Fibonacci systems exhibit four main subbands, which are successively fragmented following a trifurcation pattern [12], as expected from the fractal nature of the spectra of Fibonacci systems.

Finite values of the non-Hermitian field ($h > 0$) eject the eigenvalues from the real axis, as seen in Fig. 1. For small values of the non-Hermitian field, eigenvalues are grouped forming closed loops, which resemble the earlier subbands (see, e.g. the case $h = 0.1$ in Fig. 1). On increasing h, more and more subgaps on the real axis disappear until all eigenvalues lie on a single loop (see, e.g. the case $h = 0.2$ in Fig. 1). It is clear that eigenvalues might trace out an ellipse for $h \to \infty$ (in fact, a circumference of radius e^h). So, this process of closing subgaps on the real axis is similar to

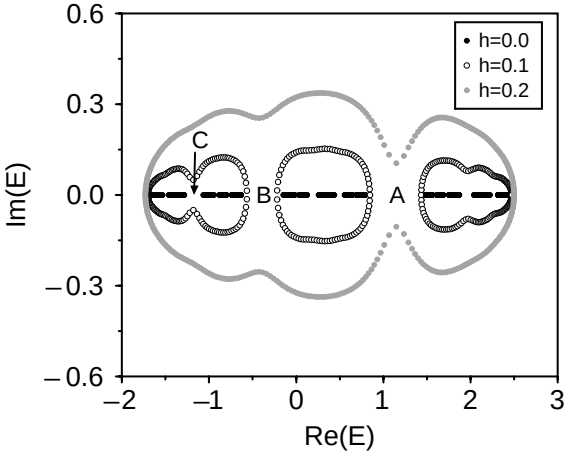

Fig. 1. Numerically obtained spectrum of the non-Hermitian Fibonacci Hamiltonian for $N = 377$ and different values of h. Labels indicate the main subgaps of the unperturbed ($h = 0$) Fibonacci system.

merging two water drops into a single one, the role of the surface tension being played by the non-Hermitian field. The larger the subgap, the larger the required h to close it. For instance, subgap C is already closed for $h = 0.1$ while subgaps A and B are not.

4. Two band model

In order to describe this process in more detail, we introduce a simple two-band model which, in spite of its simplicity, turns out to be quantitatively accurate. In this model, one focus on the subgap and the two adjacent subbands and neglect the effects of the remaining subbands. To this end, we consider a binary and infinite periodic lattice with site energies ε_e and ε_o. Taking a Bloch wave of the form $\psi_{2n}^{(k)} = U_e \exp(i2nk)$ and $\psi_{2n+1}^{(k)} = U_o \exp[i(2n+1)k]$, U_e and U_o being constants, Eq. (1) yields the following eigenenergies

$$(E_k - \varepsilon_e)(E_k - \varepsilon_o) = 4\cosh^2(ik + h). \tag{2}$$

The two solutions of the above equation correspond to the two bands of the model. The band edges of the two bands at $k = \pi/2$ are then easily calculated, so finally the subgap on the real axis is found to be

$$E_g(h) = \sqrt{E_{g0}^2 - 16\sinh^2 h}, \tag{3}$$

where $E_{g0} = |\varepsilon_e - \varepsilon_o|$ is the subgap of the Hermitian spectrum ($h = 0$). Consequently, the subgap is closed as soon as $\sinh h > E_{g0}/4$. Fig. 2 compares the numerically obtained subgaps A and B in Fig. 1 with the two-band model prediction. This naive model provides surprisingly good results in spite of its simplicity.

5. Critical to extended crossover

As mentioned in Section 1, without non-hermiticity ($h = 0$) eigenfunctions are critical but it might be delocalized in the opposite limit ($h \to \infty$). To characterize the spatial extend of eigenfunctions, Canisius and van Hemmen introduced the inverse participation ratio (IPR) [13].

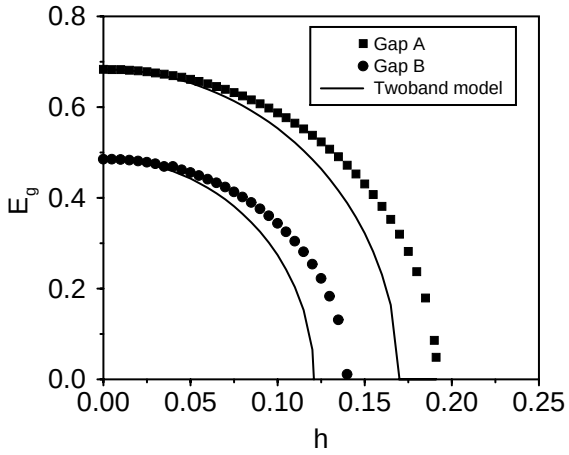

Fig. 2. Numerically obtained subgaps as a function of h spectrum of the non-Hermitian Fibonacci Hamiltonian for $N = 377$, labeled according to Fig. 1. Solid lines indicate the two-band model prediction.

Such a ratio, defined as

$$\mathrm{IPR}(E_k) = \frac{\sum_{n=1}^{N} |\psi_n^{(k)}|^4}{(\sum_{n=1}^{N} |\psi_n^{(k)}|^2)^2}, \tag{4}$$

has become a very useful magnitude in order to elucidate the spatial extend of eigenfunctions. From the general expression (4) one can see that extended states present small values of the IPR, of the order of $1/N$, whereas localized present larger values (in fact, the IPR becomes unity when the eigenfunction is localized at a single site).

Fig. 3 shows the numerically obtained IPR as a function of $\mathrm{Re}(E)$ for the non-Hermitian Fibonacci Hamiltonian with $N = 1597$ and two different values of the non-Hermitian field. Without non-hermiticity ($h = 0$), the IPR presents a number of spikes, indicating the multifractal nature of the eigenfunctions. When all subgaps are closed ($h = 0.2$, see Fig. 1), the IPR is rather smooth and close to the value $1/1597 \sim 0.0063$, suggesting the fairly extended character of the eigenfunctions. To get a clearer picture of the crossover from critical to extended eigenfunctions on increasing the non-Hermitian field, we have also calculated the mean value of the IPR over all eigenstates. This magnitude provide an estimation of the mean

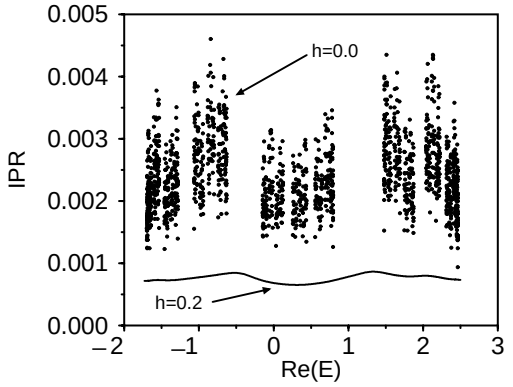

Fig. 3. IPR as a function of Re(E) of the non-Hermitian Fibonacci Hamiltonian for $N = 1597$ and two different values of the non-Hermitian field.

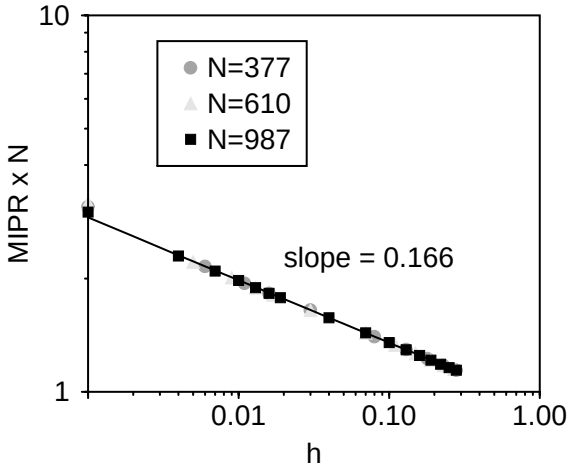

Fig. 4. Mean IPR times N as a function of non-Hermitian field h for several system length. Notice the collapse of all curves onto a single one. Solid line is a power-law fit.

degree of localization of eigenfunctions. Fig. 4 shows the collapse of the mean IPR (MIPR) times N as a function of non-Hermitian field h for several systems lengths, where it is clear that this magnitude does no reflect an abrupt transition from critical to extended states.

6. Conclusions

In conclusion, we have presented a numerical study of non-Hermitian aperiodic Hamiltonians, where diagonal terms follow a binary Fibonacci sequence. For a finite value of the non-Hermitian field ($h > 0$), the complex eigenvalues are grouped in closed loops in the complex plane. Subgaps on the real axis, being a signature of the underlying aperiodic order, are closed in a hierarchical manner on increasing the non-Hermitian field, namely the smaller the subgap, the smaller the value of the non-Hermitian field required to close it. Thus, different loops merge into a single, larger one for high enough h. A simple two-band model qualitatively explains this behavior. Finally, we found no signatures of an abrupt transition from critical to delocalized eigenfunctions through the mean IPR.

Acknowledgements

The author thanks A. Sedrakyan for bringing the attention to non-Hermitian random Hamiltonians. This work was supported by DGI under Project MAT2000-0734.

References

[1] N. Hatano, D.R. Nelson, Phys. Rev. Lett. 77 (1996) 570.
[2] N. Hatano, D.R. Nelson, Phys. Rev. B 56 (1997) 8651.
[3] N. Hatano, Physica A 254 (1998) 317.
[4] I.Y. Goldsheid, B.A. Khoruzhenki, Phys. Rev. B 80 (1998) 2987.
[5] B. Derrida, J.L. Jacobsen, R. Zeitak, J. Stat. Phys. 98 (2000) 31.
[6] J. Heinrichs, Phys. Rev. B 63 (2001) 165108.
[7] N.F. Mott, W.D. Twose, Adv. Phys. 10 (1961) 107.
[8] J. Feinberg, A. Zee, Phys. Rev. E 59 (1999) 6433.
[9] E. Maciá, F. Domínguez-Adame, Electrons, Phonons and Excitons in Low-Dimensional Aperiodic Systems, Complutense University, Madrid, 2000.
[10] E. Maciá, F. Domínguez-Adame, Phys. Rev. Lett. 76 (1996) 2957.
[11] A. Zee, Physica A 254 (1998) 300.
[12] Y. Liu, R. Riklund, Phys. Rev. B 35 (1987) 6034.
[13] J. Canisius, J.L. van Hemmen, J. Phys. C 18 (1985) 4873.