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A relativistic interaction without Klein paradox

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We consider a new type of interaction in the Dirac equation, obtained replacing $m \to m + i\gamma^0\gamma^1 V(x_1)$ in the free-particle Hamiltonian. In contrast to electrostatic-like potentials, confining potentials of the form $V(x_1) = K|x_1|^\nu$, where K and ν are positive parameters, can bind particles. This result is related to the absence of the Klein paradox.

It is well established that the Dirac equation for electrostatic linear potentials presents no bound states [1,2]. The same result is also found for arbitrary potentials increasing with increasing the separation. Galic [3] has identified this somewhat surprising result as another fine example of the Klein paradox [4]. The classic example used to illustrate the Klein paradox is the potential step [5,6]. When the potential exceeds the value $E+m$, the reflection coefficient exceeds unity. In view of the Dirac hole theory, the strong electrostatic potential raises the energy of the occupied negative energy levels so that particles tunnel through to positive energy levels in the free-field region [7], leading to a reflection coefficient larger than unity.

In a recent paper, Moshinsky and Szczepaniak [8] have considered a new type of linear interaction in the Dirac equation. The resulting equation can support bound states, In the nonrelativistic limit their equation corresponds to a three-dimensional isotropic harmonic oscillator with a strong spin–orbit coupling, so the authors gave the name of Dirac oscillator to this system. A one-dimensional version of the Dirac oscillator is obtained replacing

$$m \to m + i\gamma^0\gamma^1 m\omega x_1 \qquad (1)$$

in the free-particle Hamiltonian [9]. Here x_1 and ω stand for the spatial coordinate and the oscillator frequency, respectively. The fact that Dirac particles remain bound by this interaction suggests that the Klein paradox is absent. One of the aims of this Letter is to show that this assumption is actually valid. On the other hand, we want to discuss the Dirac equation with a new type of interaction, which will be introduced by means of the substitution

$$m \to m + i\gamma^0\gamma^1 V(x_1) , \qquad (2)$$

where $V(x_1)$ is an arbitrary, time-independent potential. In the special case $V(x_1) = m\omega x_1$ (2) reduces to (1). Therefore the Dirac equation reads

$$[p_\mu\gamma^\mu - m - i\gamma^0\gamma^1 V(x_1)]\psi = 0 . \qquad (3)$$

In order to solve (3) we use the ansatz

$$\psi = [p_\mu\gamma^\mu + m - i\gamma^0\gamma^1 V(x_1)]\chi , \qquad (4)$$

so that, after substituting (4) into (3), we obtain

$$[p_\mu p^\mu - m^2 - V^2(x_1) - ip_\mu\gamma^\mu\gamma^0\gamma^1 V(x_1) - i\gamma^0\gamma^1 V(x_1) p_\mu\gamma^\mu]\chi = 0 . \qquad (5)$$

Since $p_0 = E$, p_2 and p_3 are constants of motion, the spinor solution χ takes the form

$$\chi = \chi(x_1) \exp[i(p_0x_0 - p_2x_2 - p_3x_3)] , \qquad (6)$$

where $\chi(x_1)$ satisfies the equation

$$[-d^2/dx_1^2 + V'(x_1)\gamma^0 + V^2(x_1) - 2i\gamma^0\gamma^1(p_2\gamma^2 + p_3\gamma^3)V(x_1)]\chi(x_1) = (E^2 - m^2 - p_\perp^2)\chi(x_1) \qquad (7)$$

and we have introduced the notation $p_\perp^2 = p_2^2 + p_3^2$. Here the prime denotes the derivative with respect

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to the argument. For the present problem we choose the representation

$$\gamma^0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \gamma^k=\begin{pmatrix}0 & \sigma^k\\ -\sigma^k & 0\end{pmatrix}. \tag{8}$$

To obtain a Schrödinger-like equation we perform the unitary transformation

$$\chi(x_1)=U\begin{pmatrix}\phi^+(x_1)\\ \phi^-(x_1)\end{pmatrix}, \tag{9}$$

where $\phi^{\pm}$ are two-component spinors and

$$U=\frac{1}{\sqrt{2p_\perp(p_\perp-p_2)}}\,\mathrm{i}\gamma^0\gamma^1 \times[(p_2-p_\perp)\gamma^2+p_3\gamma^3]\,. \tag{10}$$

Thus we find

$$[-\mathrm{d}^2/\mathrm{d}x_1^2\pm V'(x_1)+V^2(x_1) \pm 2p_\perp V(x_1)\sigma^3]\phi^{\pm}(x_1) =(E^2-m^2-p_\perp^2)\phi^{\pm}(x_1)\,. \tag{11}$$

Let $\phi^{\pm}$ be eigenvectors of σ^3 corresponding to the eigenvalue $s^{\pm}$ (1 or -1). Thus we finally write

$$[-\mathrm{d}^2/\mathrm{d}x_1^2\pm V'(x_1)+V^2(x_1) \pm 2p_\perp V(x_1)s^{\pm}]\phi^{\pm}(x_1) =(E^2-m^2-p_\perp^2)\phi^{\pm}(x_1)\,. \tag{12}$$

This is nothing but a Schrödinger-like equation for the two-component spinors $\phi^{\pm}$. The complete solution of the Dirac equation (3) is found by means of (4), (6) and (9) in a straightforward way, provided that $\phi^{\pm}$ are known through eq. (12). Note that (12) reduces to a nonrelativistic harmonic oscillator equation in the case $V(x_1)=m\omega x_1$.

Let us consider a confining type potential of the form $V(x_1)=K|x_1|^{\nu}$, where K and ν are positive parameters. For large values of the spatial coordinate x_1, the asymptotic form of (12) is simply written as

$$(-\mathrm{d}^2/\mathrm{d}x_1^2+K^2x_1^{2\nu})\phi^{\pm}(x_1) =(E^2-m^2-p_\perp^2)\phi^{\pm}(x_1)\,. \tag{13}$$

Therefore, one clearly sees that an infinite set of bound states occurs along the x_1 direction. Hence eq. (3) can also be used to explain the observed confinement of quarks. The form of the interaction is not manifestly covariant, although kinematics terms are. Concerning the three-dimensional Dirac oscillator, Benítez et al. [10] have recently demonstrated that the interaction term can be written in a covariant form. Their arguments could be extended to include more general interactions of the form $\mathrm{i}\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{r}f(|\boldsymbol{r}|)$, f being an arbitrary function. The one-dimensional version of this kind of interaction is nothing but what appears in eq. (2). Therefore the existence of bound states is frame independent, as expected from a physical point of view. The fact that the occurrence of bound states does not depend on the particular frame selected is also valid for two Dirac particles interacting by means of potentials similar to that given in (1), in the instant form approximation [11].

The occurrence of bound states is due to the fact that the interaction we have introduced in (3) presents no Klein paradox, as we mentioned above. To demonstrate this point, we now take a potential step of the form $V(x_1)=V_0\vartheta(x_1)$, ϑ being the Heaviside step function and $V_0>0$. For particles moving along the x_1 direction ($p_2=p_3=0$), a straightforward calculation yields the following reflection coefficient,

$$R=\left|\frac{(E^2-m^2)^{1/2}-(E^2-m^2-V_0^2)^{1/2}+\mathrm{i}V_0}{(E^2-m^2)^{1/2}+(E^2-m^2-V_0^2)^{1/2}-\mathrm{i}V_0}\right|^2. \tag{14}$$

It is an easy matter to check that $R\leqslant 1$ for all values of V_0 ranging from 0 to ∞. Therefore, particles cannot tunnel from states below V_0 on the right side to the left. Hence we can conclude that there is no Klein paradox in this case. The absence of this tunneling mechanism is clearly related to how the product $\mathrm{i}\gamma^0\gamma^1$ works in the Dirac equation (3). The effects of the term $\mathrm{i}\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{r}$, corresponding to the three-dimensional Dirac oscillator, has been studied by Martínez et al. [12]. These authors found that the supersymmetric properties of the Dirac oscillator Hamiltonian are related to the stability of the Dirac sea. Although these authors did not discuss the Klein paradox, such a stability would explain the absence of tunneling for the ϑ barrier. From a more heuristic point of view, Domínguez-Adame and González [9] suggest that the possible origin of the Dirac oscillator interaction is related to scalar potentials rather than electrostatic potentials, which are known not to present the Klein paradox. The extension of these results to the one-dimensional interaction (2) is straightforward.

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References

[1] R.K. Su and Z. Yuhong, J. Phys. A 17 (1984) 851.
[2] R.K. Su and Z.Q. Ma, J. Phys. A. 19 (1986) 1739.
[3] H. Galic, Am. J. Phys. 56 (1988) 312.
[4] O. Klein, Z. Phys. 53 (1929) 157.
[5] J.D. Bjorken and S.D. Drell, Relativistic quantum mechanics (McGraw-Hill, New York, 1964) p. 40.
[6] S. Flügge, Practical quantum mechanics, Vol. II (Springer, Berlin, 1974) p. 213.
[7] B.H.J. McKellar and G.J. Stephenson, Phys. Rev. A 36 (1987) 2566.
[8] M. Moshinsky and A. Szczepaniak, J. Phys. A 22 (1987) L817.
[9] F. Domínguez-Adame and M.A. González, Europhys. Lett. 13 (1990) 193.
[10] J. Benítez, R.P. Martínez y Romero, H.N. Núñez Yépez and A.L. Salas-Brito, Phys. Rev. Lett. 64 (1990) 1643.
[11] F. Domínguez-Adame and B. Méndez, Can. J. Phys. 69 (1991) 780.
[12] R.P. Martínez y Romero, M. Moreno and A. Zentella, Phys. Rev. D 43 (1991) 2036.