The transitions to chaos in the physics of condensed matter and complex systems

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Outline

- Introduction
- Localization
- Zipf’s and Benford’s laws
- Renewal processes
- Central limit theorem
- Summary

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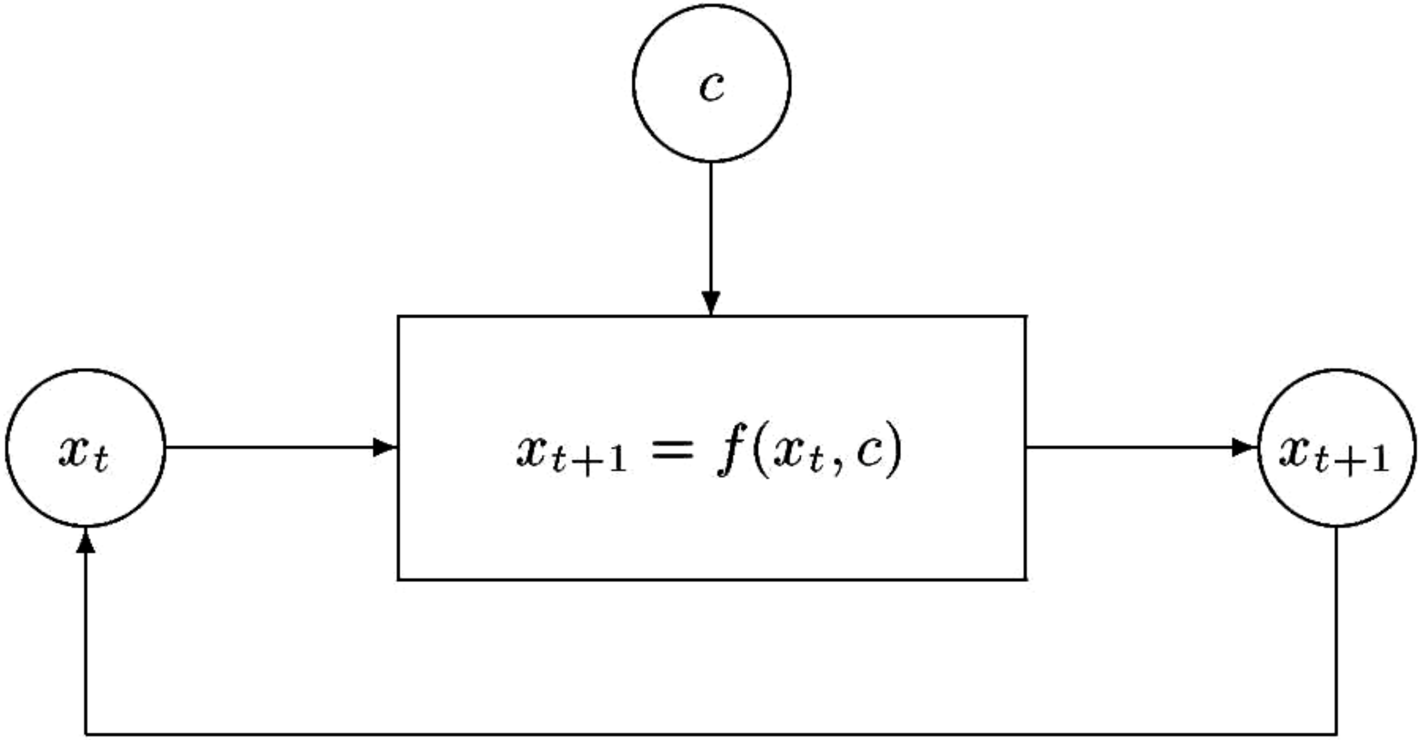
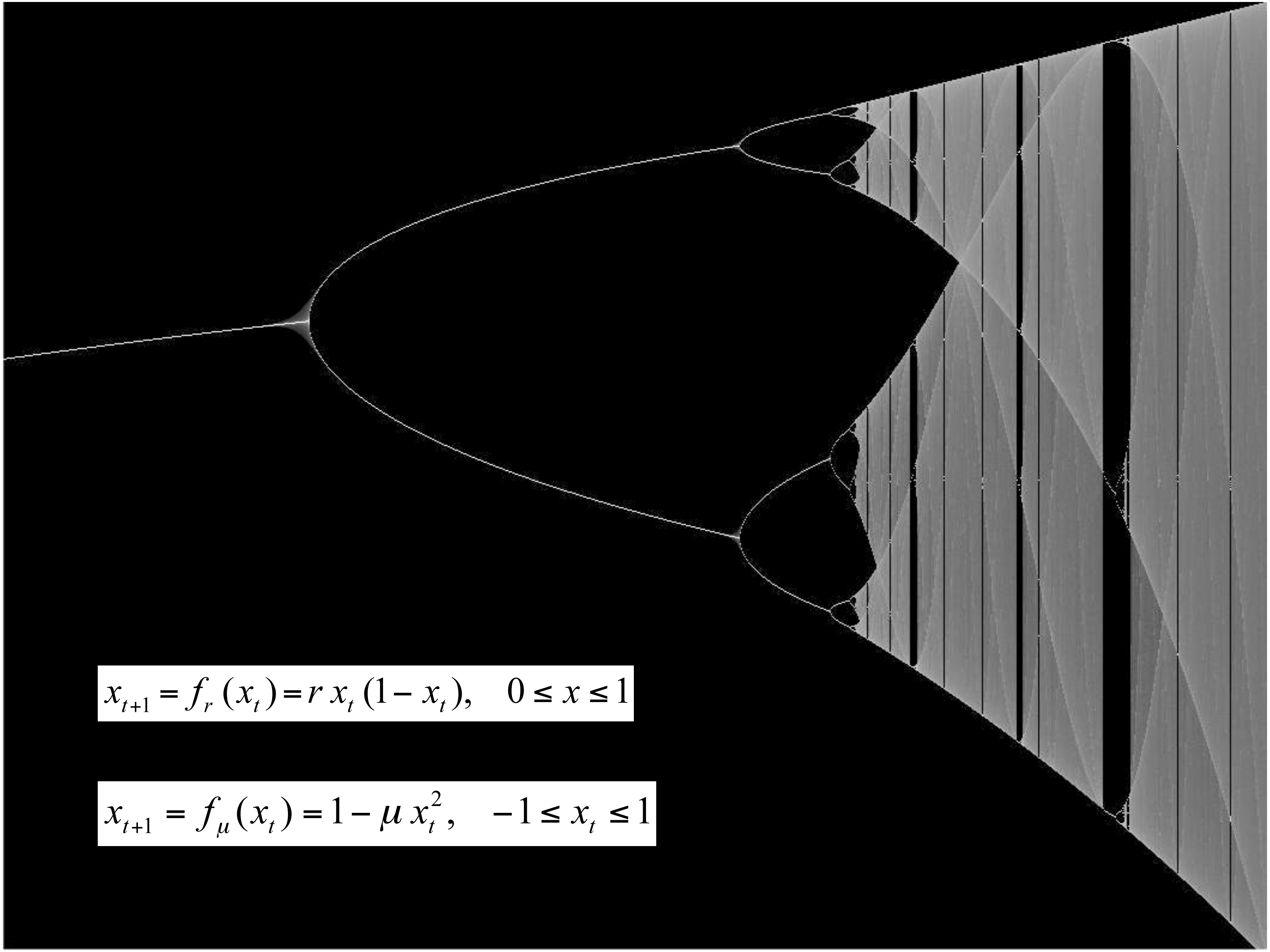
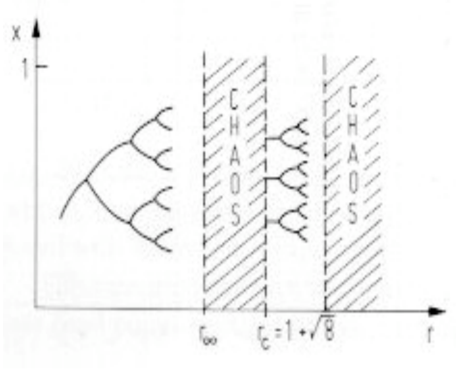

c
x_t
$x_{t+1} = f(x_t, c)$
x_{t+1}

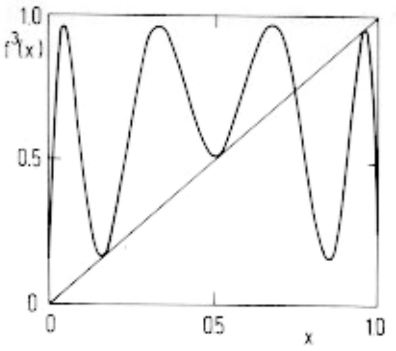


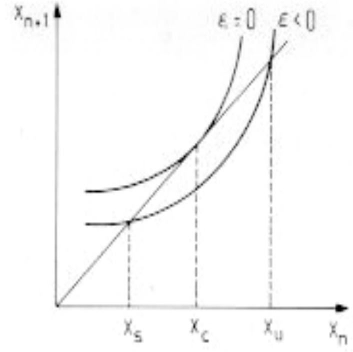
$$x_{t+1} = f_r(x_t) = r\,x_t\,(1 - x_t), \quad 0 \le x \le 1$$

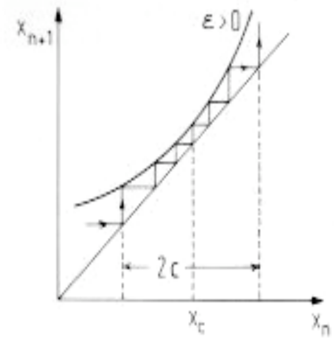
$$x_{t+1} = f_\mu(x_t) = 1 - \mu\,x_t^2, \quad -1 \le x_t \le 1$$

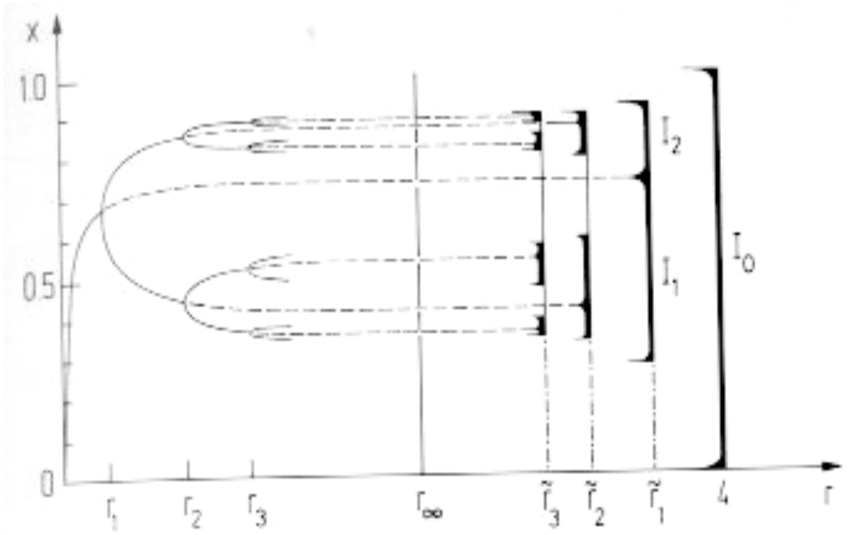
Intermittency and band splitting











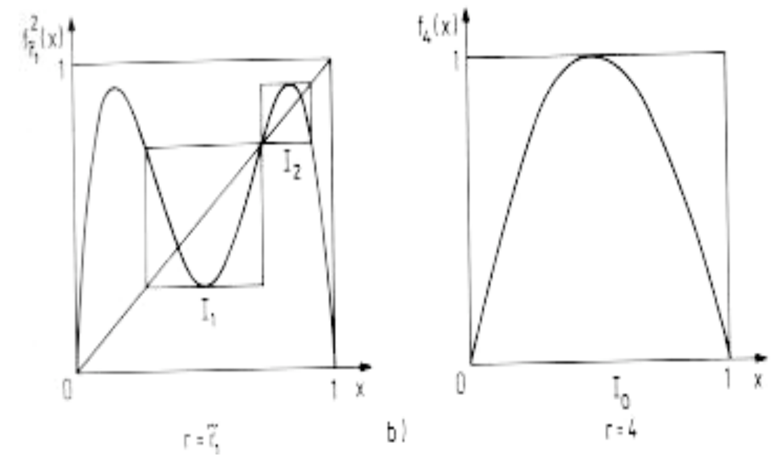


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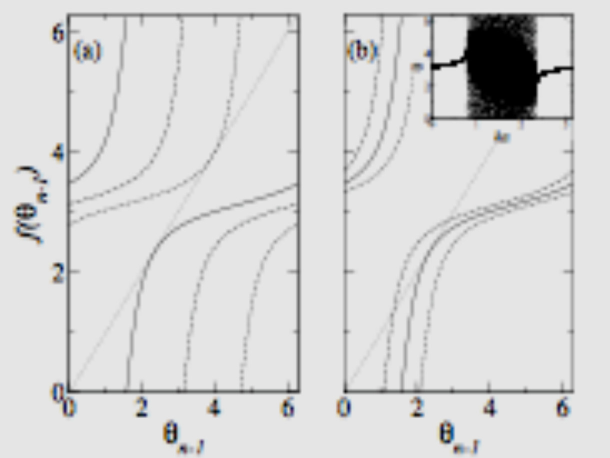
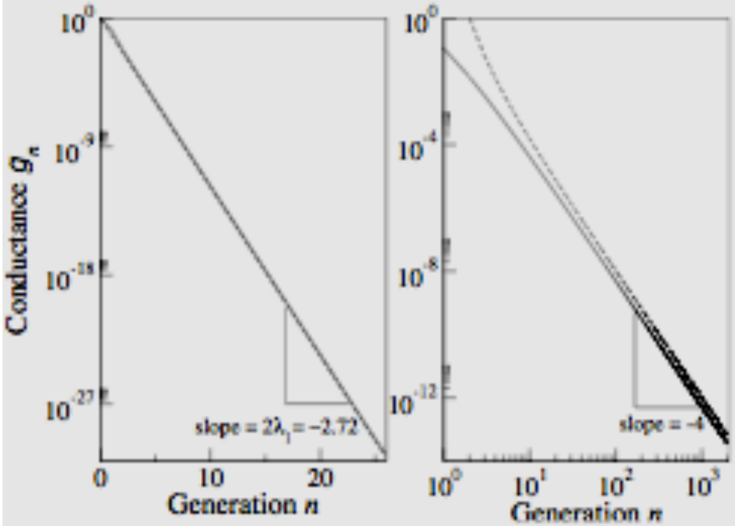
FIG. 1. A double Cayley tree of connectivity $K=2$ and lattice constant a. Each bond is a perfect one-dimensional conductor.

$$f(\theta_n) = 2ka - \theta_n + \arctan\left(\frac{\sin\theta_n + \sqrt{1-2\varepsilon}\sin 2ka}{\cos\theta_n - \sqrt{1-2\varepsilon}\cos 2ka}\right)$$



$$\xi_n = \exp_q(\lambda_q n) \equiv \left[1-(q-1)\lambda_q n\right]^{-1/(q-1)}$$

$$g_n \propto \left(1 - \frac{1}{2}\lambda_{3/2} n\right)^{-4}$$



PHYSICAL REVIEW E **80**, 045201(R) (2009)

Equivalence between the mobility edge of electronic transport on disorderless networks and the onset of chaos via intermittency in deterministic maps

M. Martínez-Mares[1] and A. Robledo[2]

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Hungarian aristocrats around 1500

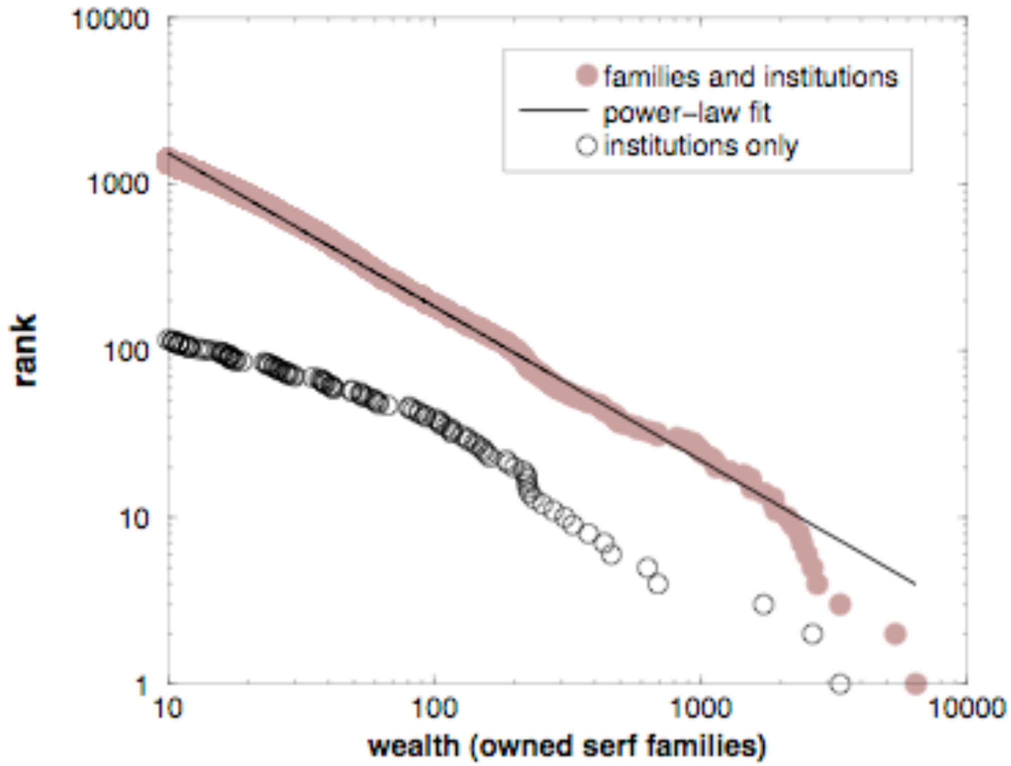
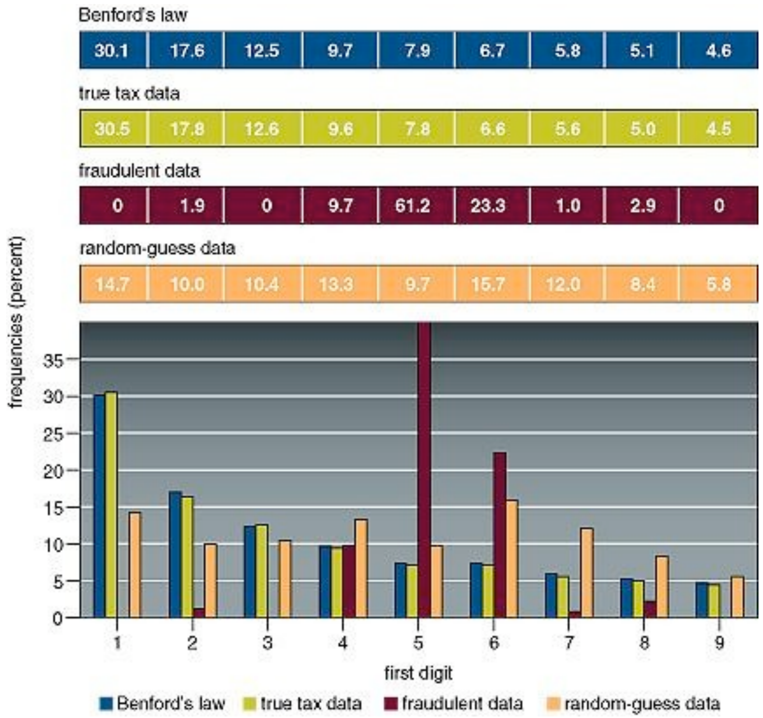


Fig. 1. The rank of the top 8% aristocrat families and institutions as a function of their estimated total wealth on a log-log scale. Measurement results for the Hungarian noble society in the year 1550. The total wealth of a family is estimated as the number of owned serf families. The power-law fit suggests a Pareto index $\alpha = 0.92$.



(From "The First-Digit Phenomenon" by T. P. Hill, American Scientist, July-August 1998)

Benford's law can be used to test for fraudulent or random-guess data in income tax returns and other financial reports. Here the first significant digits of true tax data taken by Mark Nigrini from the lines of 169,662 IRS model files follow Benford's law closely. Fraudulent data taken from a 1995 King's County, New York, District Attorney's Office study of cash disbursement and payroll in business do not follow Benford's law. Likewise, data taken from the author's study of 743 freshmen's responses to a request to write down a six-digit number at random do not follow the law. Although these are very specific examples, in general, fraudulent or concocted data appear to have far fewer numbers starting with 1 and many more starting with 6 than do true data.

We have built on the work of Pietronero, Tossi, Tossi & Vespigniani (2001) and:

- Found a likely statistical-mechanical structure underlying the laws of Zipf and Benford

- Discovered a complete analogy with the transition to chaos via intermittency

- Altamirano, C. , Robledo, A.,

"Generalized thermodynamics underlying the laws of Zipf and Benford"

Lecture Notes of the Institute for Computer Sciences, Social-Informatics and Telecommunications Engineering (LNICST)

A series of Springer-Verlag, Vol. 5 (Complex Sciences), pp. 2232-2237 (2009).

Some calculations

- Benford's law corresponds to a uniform distribution in logarithmic space

$$\int P(\log N)d(\log N) = C\int N^{-1}dN \quad \Rightarrow \quad P(N) \sim N^{-1}$$

- The probability of observation of the first digit *n* of number *N* is given by

$$p(n) = \int_{n}^{n+1} N^{-1}\, dN = \int_{n}^{n+1} d(\log N) = \log((n+1)/n)$$

- The rank *k* of a set of $\mathcal{N}$ data numbers is

$$k = \mathcal{N} \int_{N(k)}^{N_{\max}} N^{-1}\, dN = \mathcal{N} \log(N_{\max}/N(k))$$

- Or, equivalently,

$$N(k) = N_{\max} \exp\left[-\mathcal{N}^{-1}\, k\right]$$

But when $P(N) \sim N^{-\alpha}, \quad \alpha > 1$

- The probability of observation of the first digit *n* of number *N* is given by

$$p(n) = \int_{n}^{n+1} N^{-\alpha}\, dN = \frac{1}{1-\alpha}\left[(n+1)^{1-\alpha} - n^{1-\alpha}\right], \quad \alpha > 1$$

- The rank *k* of a set of $\mathcal{N}$ data numbers is

$$k = \mathcal{N} \int_{N(k)}^{N_{\max}} N^{-\alpha}\, dN = \frac{1}{1-\alpha}\left[N_{\max}^{1-\alpha} - N(k)^{1-\alpha}\right], \quad \alpha > 1$$

- In terms of the *q*-deformed logarithmic and exponential functions $\ln_q x \equiv \dfrac{x^{1-q} - 1}{1-q}$

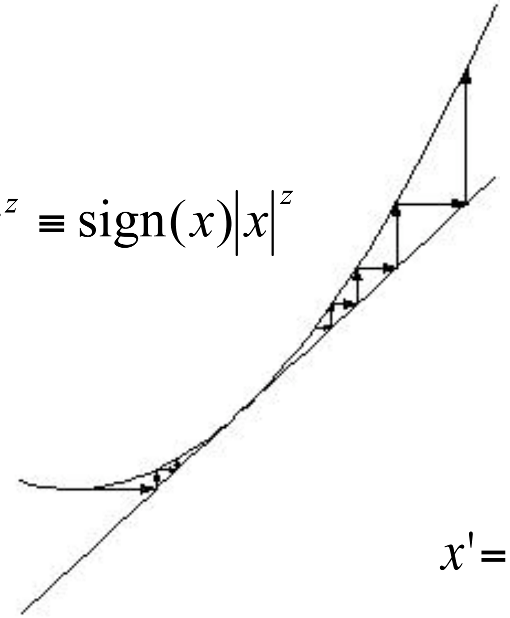
and $\exp_q(x) \equiv \left[1 + (1-q)x\right]^{1/1-q}$ we have

$$\boxed{\log_\alpha N(k) = \log_\alpha N_{\max}(\mathcal{N}) - \mathcal{N}^{-1}\, k} \qquad \text{or} \qquad N(k) = N_{\max} \exp_\alpha\left[-N_{\max}^{\alpha-1}\, \mathcal{N}^{-1}\, k\right]$$

- In the limit $N_{\max} \to \infty$ we recover Zipf's law

$$N(k) \propto k^{1/1-\alpha}, \quad \alpha > 1$$

Hu & Rudnick (1982) RG fixed-point map for tangent bifurcation



$$x' = f^{(n)}(x) = x + u x^{z} + \ldots, z > 1$$

$$x^{z} \equiv \mathrm{sign}(x)|x|^{z}$$

$$f^{*}(f^{*}(x)) = \gamma^{-1} f^{*}(\gamma x)$$

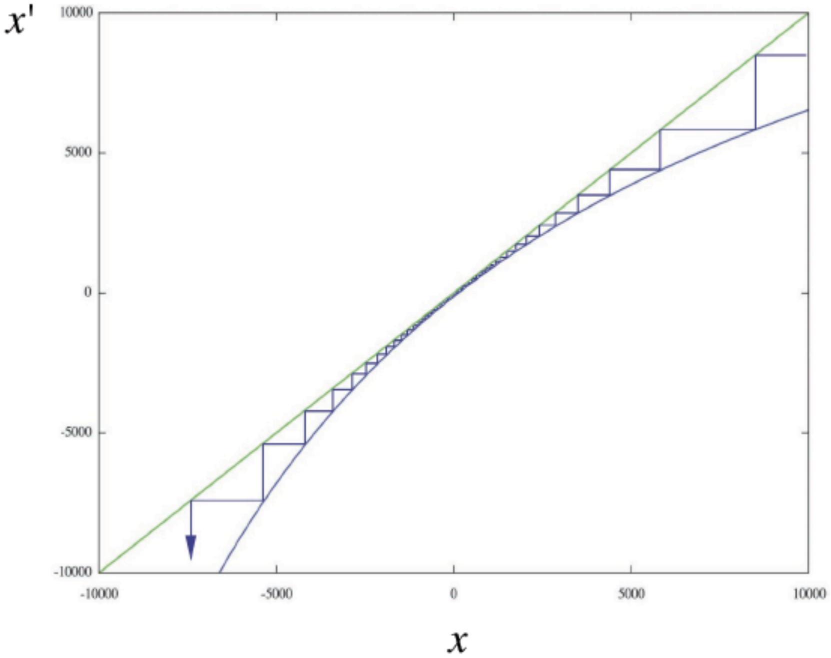
$$x'^{-(z-1)} = x^{-(z-1)} - (z-1)u, \quad \gamma = 2^{1/(z-1)}$$

$$x' = f^{*}(x) = x \exp_{z}(u x^{z-1}) \equiv x\left[1-(z-1)u x^{z-1}\right]^{-\frac{1}{z-1}}$$

$$x_t^{-(z-1)} = x_0^{-(z-1)} - (z-1)ut$$

$$\boxed{\ln_z x_t = \ln_z x_0 - ut, \quad x_0 > 0}$$

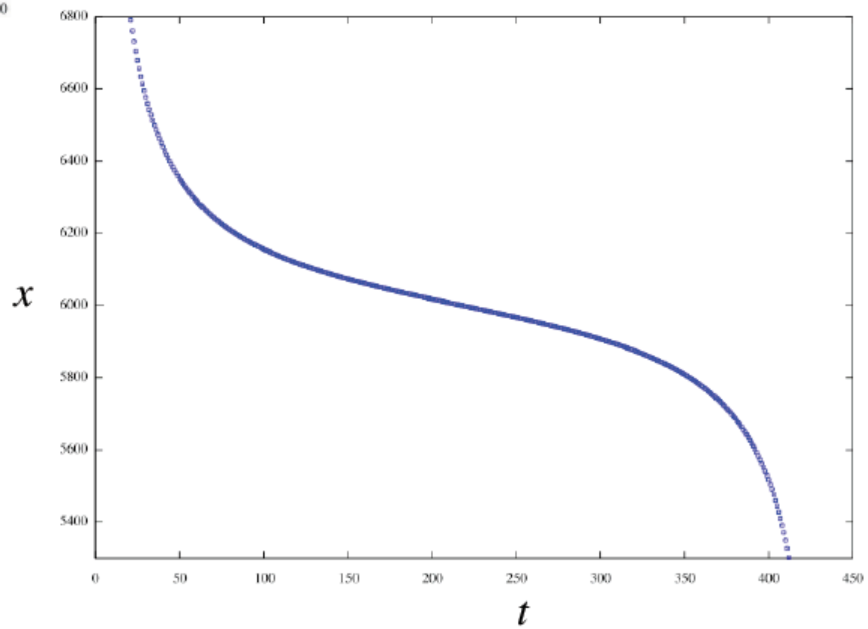
$$x_t = f^{*(t)}(x_0) = x_0 \exp_z(u x_0^{z-1}\, t) = x_0[1-(z-1)u x_0^{z-1} t]^{-\frac{1}{z-1}}$$



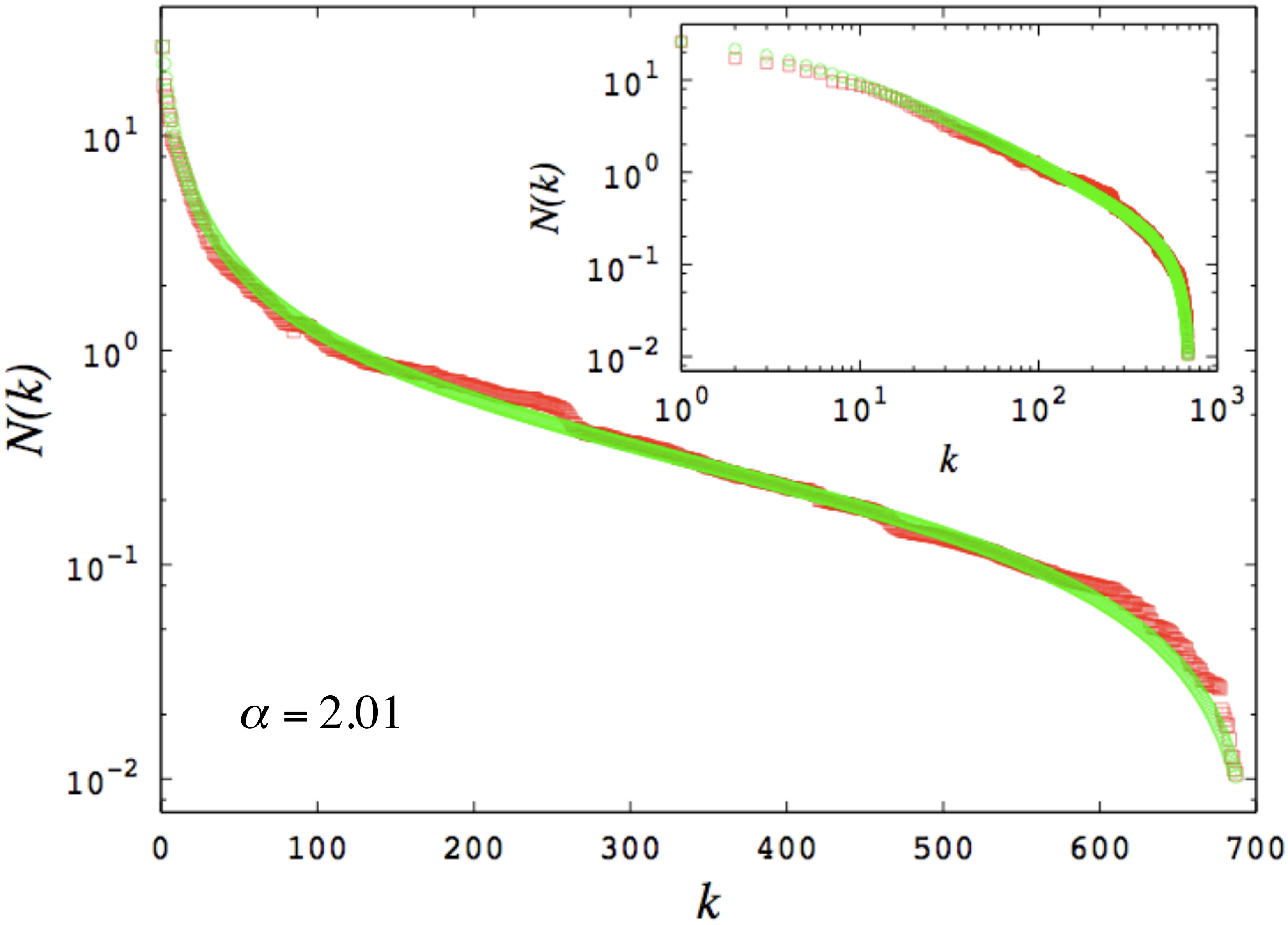
Map off tangency

$$x' = f^*(x) + \varepsilon = x\exp_z(ux^{z-1}) + \varepsilon, \quad \varepsilon << 1$$

and its trajectories



Eigenfactors



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• We offer a statistical-mechanical interpretation of the basic elements that constitute the theory of renewal processes.

• Our purpose for developing this analogy is to facilitate the application of techniques and approximations built up and tested through a large amount of studies of thermal systems.

• Potentially, these methodologies may have an effect in the study of complex systems in a variety of fields, in ecology, economy, sociology, etc., where stochastic processes such as that for the renewing of events often arise.

Statistical-mechanical structure for renewal stochastic processes

Jorge Velázquez and Alberto Robledo

JSTAT_041P_0110 v1

Basics of renewal processes

$\psi(t)$ density distribution for waiting times

$$\psi_n(t) = \int_0^t dt'\, \psi(t-t')\, \psi_{n-1}(t'), \quad n \geq 1$$

$$\hat{\psi}_n(\varepsilon) = \left[\hat{\psi}(\varepsilon)\right]^n$$

$$\psi(t;z) \equiv \sum_{n=1}^{\infty} \psi_n(t)\, z^n$$

$$\hat{\psi}(\varepsilon;z) = \sum_{n=1}^{\infty} \hat{\psi}_n(\varepsilon)\, z^n = \hat{\psi}(\varepsilon)\, z \left[1 - \hat{\psi}(\varepsilon)\, z\right]^{-1}$$

$\hat{\psi}(\varepsilon;z)$ everything can be extracted from this function

An illustration

$$\psi(t) = b\exp(-bt)$$

exponential waiting-time distribution

$$\hat{\psi}(\varepsilon) = b(b+\varepsilon)^{-1}$$

$$\hat{\psi}(\varepsilon;z) = bz\left(b+\varepsilon-bz\right)^{-1}$$

$$\hat{\psi}_n(\varepsilon) = b^n(b+\varepsilon)^{-n}$$

$$\psi(t;z) = btz\,\exp(1-bt+btz)$$

$$\psi_n(t) = \frac{(bt)^n}{n!}\exp(-bt)$$

Poisson distribution

A statistical-mechanical structure?

$$\hat{\psi}(\varepsilon;z) = \sum_{n=1}^{\infty} \hat{\psi}_n(\varepsilon)\, z^n, \quad z = \exp\mu$$

a grand canonical partition function?

$$\hat{\psi}_n(\varepsilon) = \int_0^{\infty} dt\, \exp(-\varepsilon t)\, \psi_n(t) = \left[\hat{\psi}(\varepsilon)\right]^n$$

a canonical partition function?

$$S_{\varepsilon,\mu} \equiv \ln\hat{\psi}(\varepsilon;z),$$
$$S_{\varepsilon,n} \equiv \ln\hat{\psi}_n(\varepsilon),$$
$$S_{t,\mu} \equiv \ln\psi(t:z),$$
$$S_{t,n} \equiv \ln\psi_n(t)$$

entropies or Massieu potentials?

$$S_{t,n} = t\varepsilon - n\mu$$

an Euler relation?

From canonical to micro canonical

$$\psi_n(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\varepsilon \exp(\varepsilon t)\,\hat{\psi}_n(\varepsilon) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\varepsilon \exp\Big[n\big(\varepsilon\tau + \ln\hat{\psi}(\varepsilon)\big)\Big], \quad \tau \equiv t/n$$

saddle-point approximation

$$n^{-1}\ln\psi_n(n\tau) = \varepsilon_0\tau + \ln\hat{\psi}(\varepsilon_0)$$

$$\tau = -\left.\frac{d}{d\varepsilon}\ln\hat{\psi}(\varepsilon)\right|_{\varepsilon=\varepsilon_0}$$

$$\psi_n(t) \cong \exp S_{t,n} \quad \text{where} \quad S_{t,n} = -t\varepsilon_0 + S_{\varepsilon_0,n} \quad \text{with} \quad S_{\varepsilon_0,n} \equiv \ln\hat{\psi}_n(\varepsilon_0)$$

when $\psi(t) = \exp(-bt)$

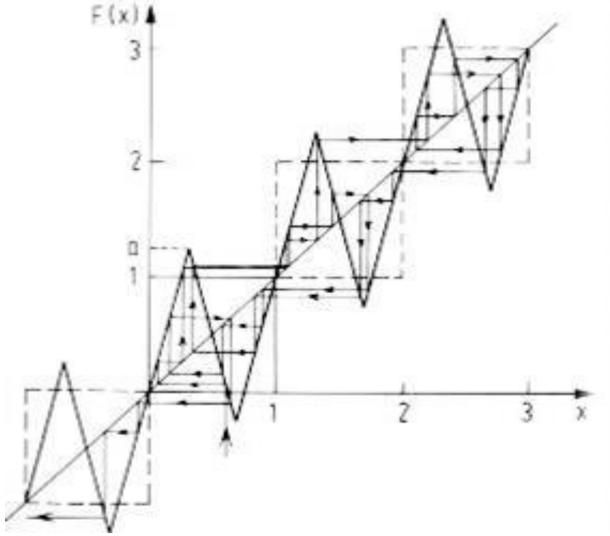
$$t = n(b+\varepsilon_0)^{-1}$$

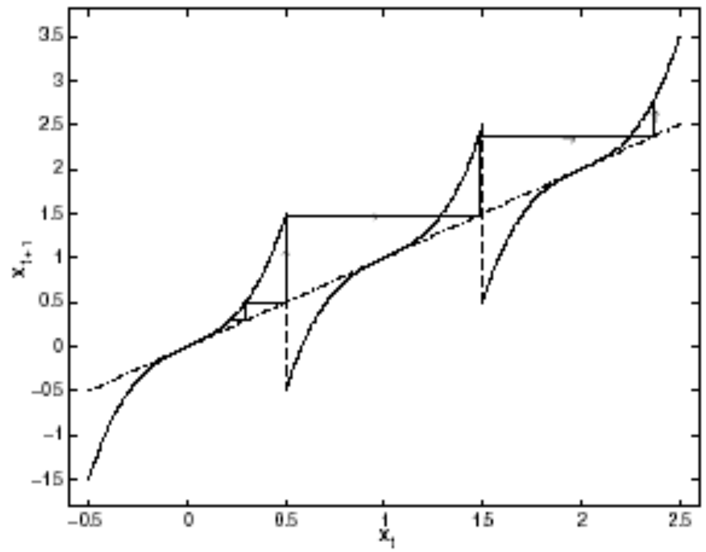
$$S_{t,n} = \ln\left[(btn^{-1})^n \exp(n)\exp(-bt)\right] \quad \Leftrightarrow \quad S_{t,n} = \ln\left[\frac{(bt)^n}{n!}\exp(-bt)\right]$$

Anomalous deterministic diffusion

- **Repeated-cell maps**

$$x_{t+1} = f(x_t), \text{ where } f(m+x) = m + f(x),\ m = \dots,-1,0,1,\dots \text{ and } f(-x) = f(x)$$





- **Distribution of cell residence time intervals**

$$\Psi(t,x_0) = \int_0^t dt'\, \psi(t',x_0)$$

BG statistics

$$\Psi(t,x_0) = \exp\left[-a_1(x_0)\,t\right]$$

***q*-statistics**

$$\Psi(t,x_0) = \exp_q\left[-a_q(x_0)\,t\right]$$

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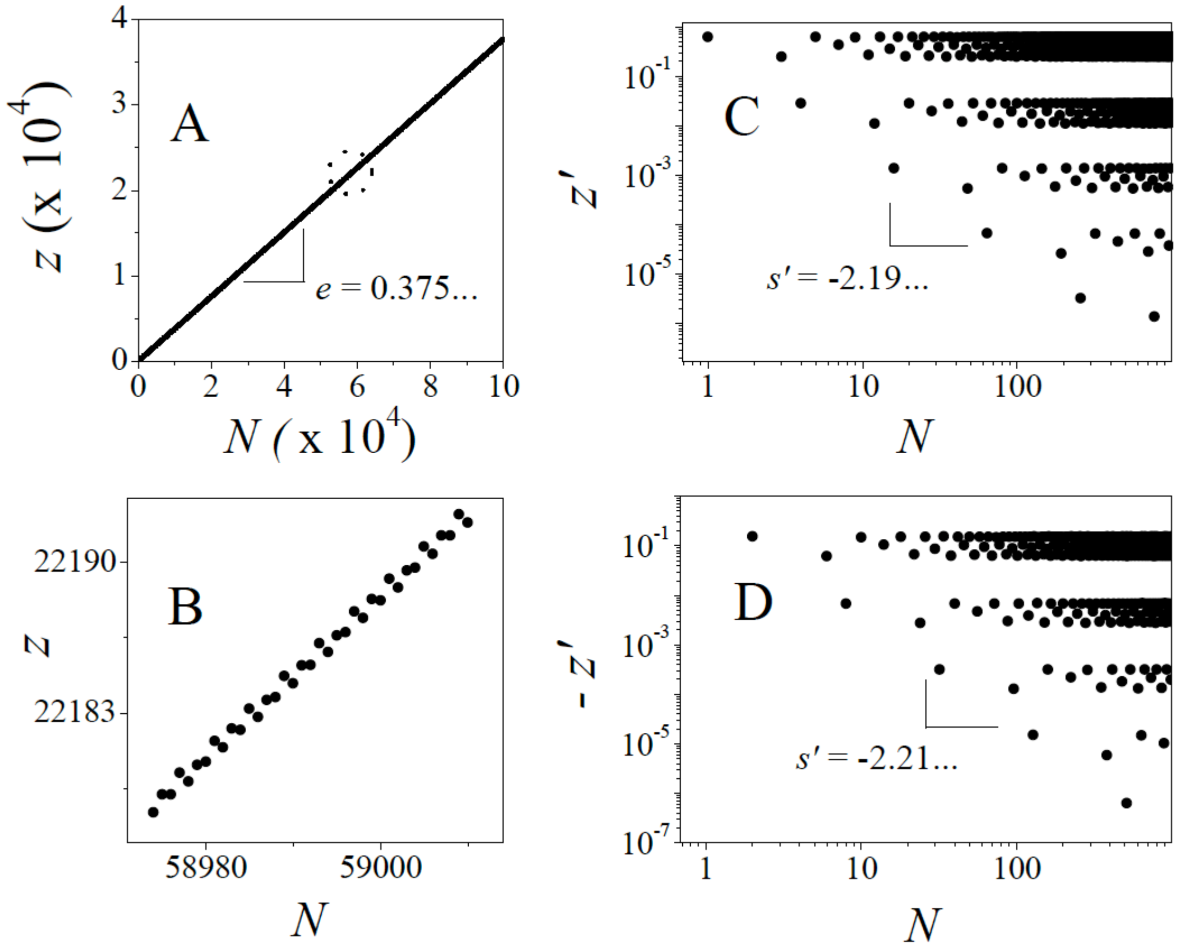
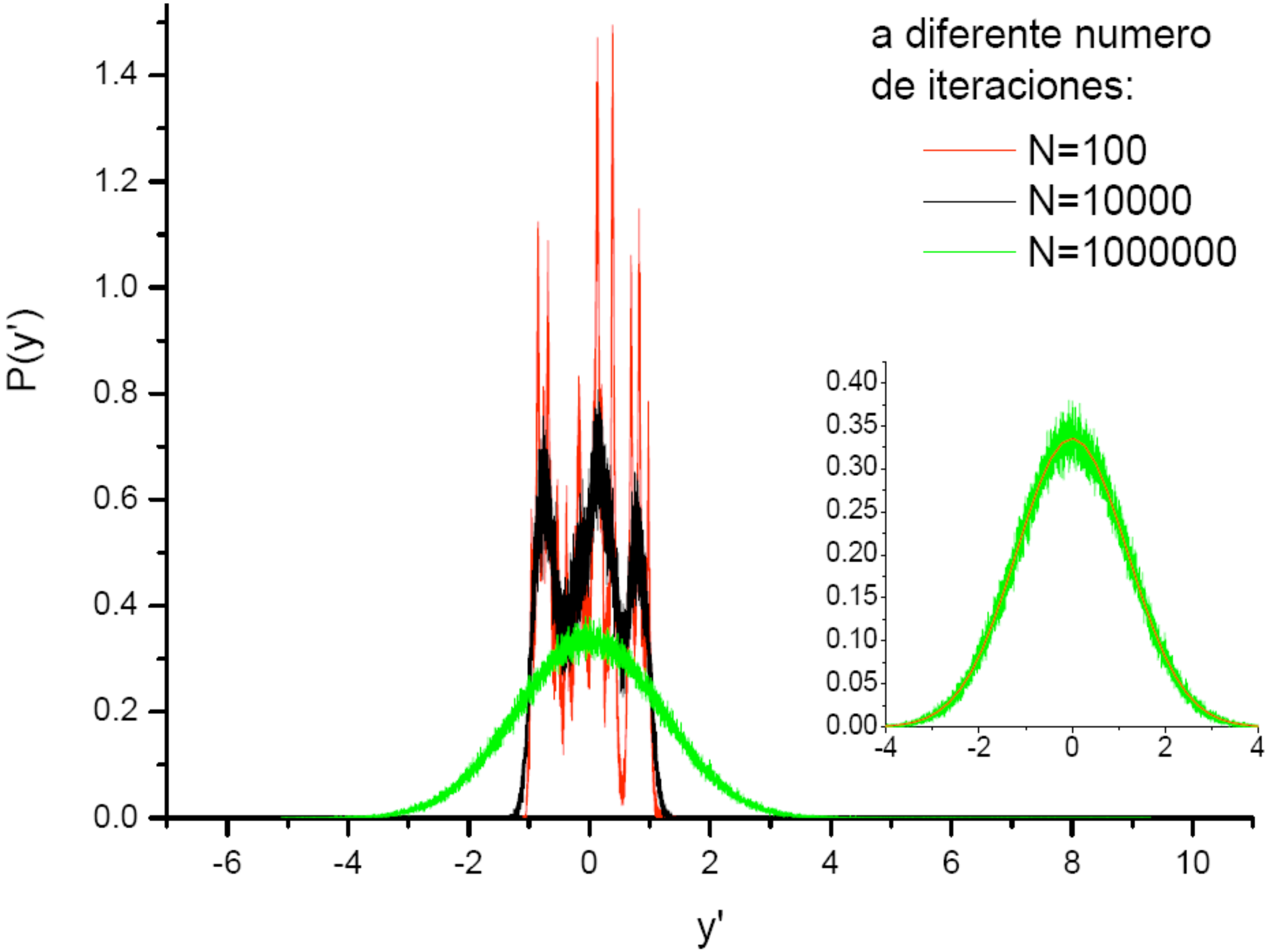
Sum of positions inside the Feigenbaum attractor

$$z_{\mu_\infty}(N) \equiv \sum_{t=1}^{N} x_t \qquad z'_{\mu_\infty}(N) \equiv \sum_{t=1}^{N} (x_t - e)$$



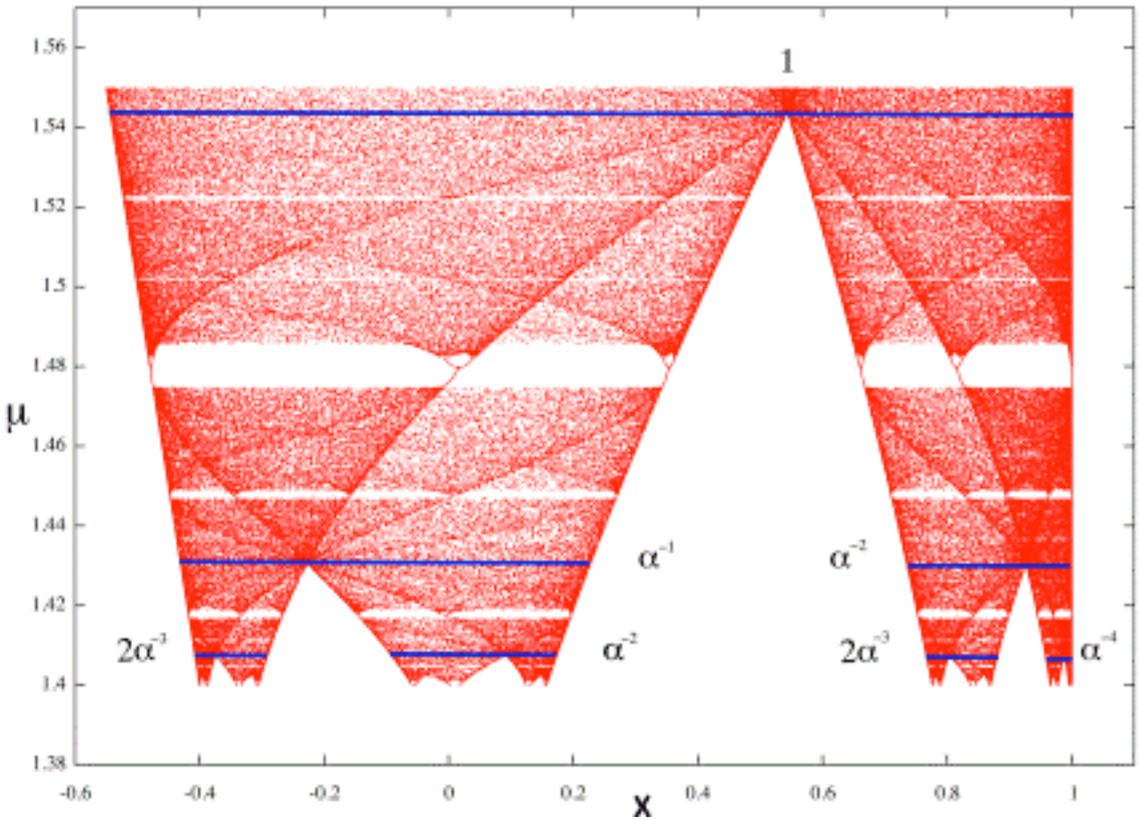
One percent deviation from the onset of chaos



μ

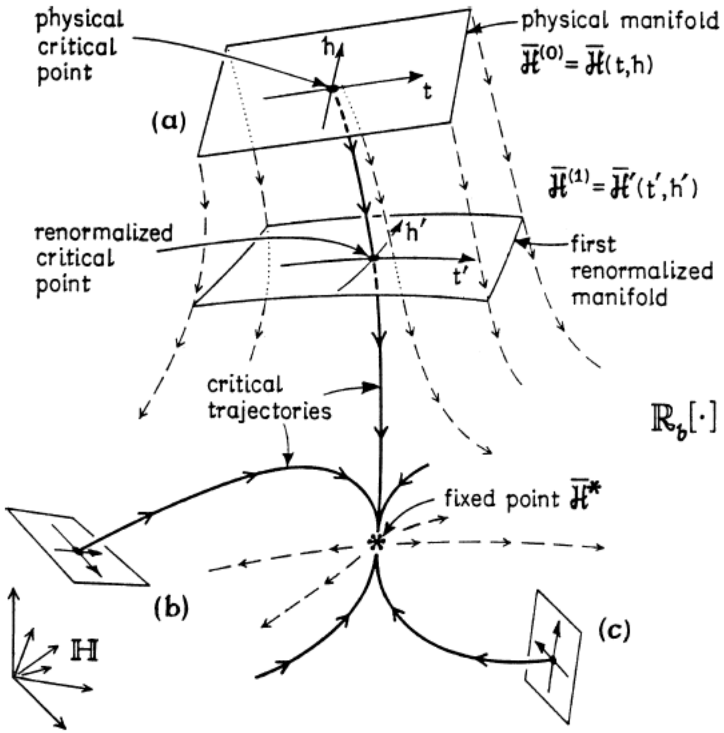
Band splitting in 'mean field'

- Consider that band splitting scales with the most crowded and sparse regions of the multifractal attractor, i.e. α^{-2k} or α^{-k}



- The widths of the bands form a Pascal triangle across band splitting

• **A RG view of the CLT as illustrated by chaotic band attractors**



Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Renormalization group structure for sums of variables generated by incipiently chaotic maps

Miguel Angel Fuentes[1,2,3] and Alberto Robledo[4]

(2010) P01001

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