Replicator dynamics for the iterated prisoner's dilemma with three types of players

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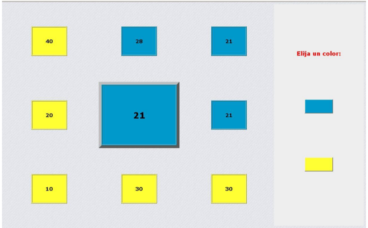
Prisoner's dilemma

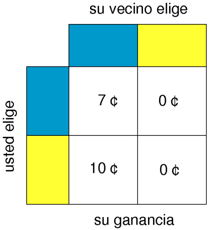
	Cooperate	Defect
Cooperate	**R**, R	**S**, T
Defect	**T**, S	**P**, P

- Temptation to defect
- Reward for mutual cooperation
- Punishment for mutual defection
- Sucker's payoff

$$T > R > P \geq S, \; 2R > T + S$$

Experiment





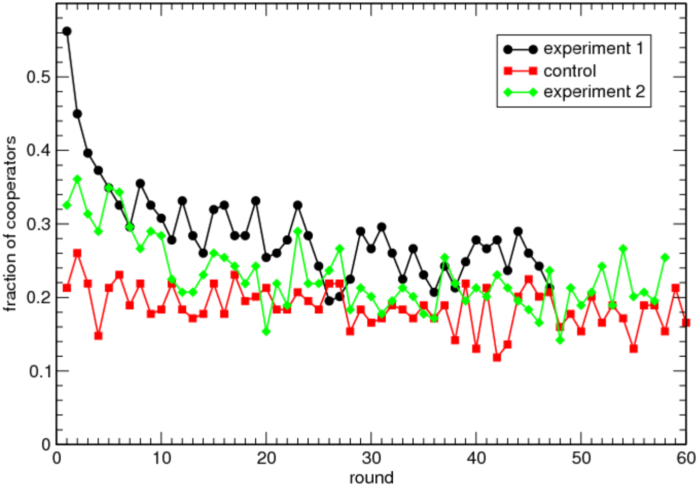
- Square lattice 13×13
- Experiment 1 $\rightarrow$ Control $\rightarrow$ Experiment 2

"Social experiments in the mesoscale: Humans playing a spatial Prisoner's Dilemma" J. G., C. Fosco, L. Araujo, JA. Cuesta, A. Sánchez

PLoS ONE 5, e13749 (2010)

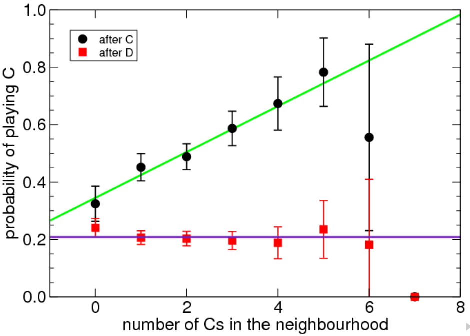
Results

- Cooperation level $\cong 25\%$

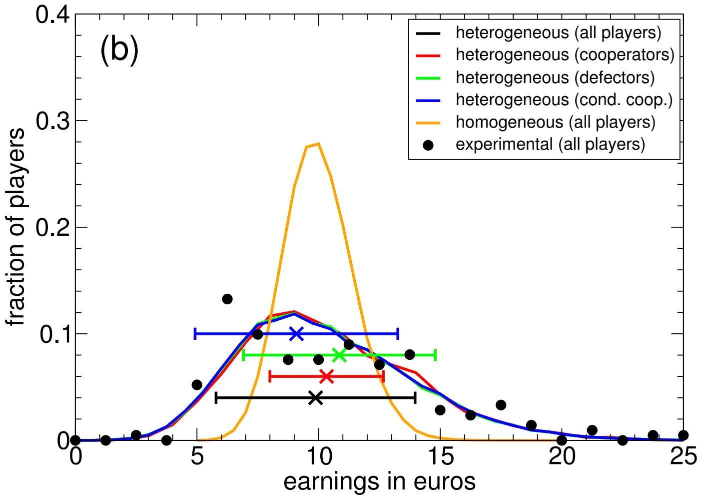


Results

Type of player	Mark	Experiment 1	Control	Experiment 2
Defectors	**D**	24%	43%	41%
Cooperators	**C**	1.8%	1.8%	5.3%
Conditional cooperators	**X**	74%	56%	54%



Earnings for different types



Idea

- Why this heterogeneity?
- Is there an evolutionary explanation?
- We start with the simplest case.
- Pairwise prisoner's dilemma

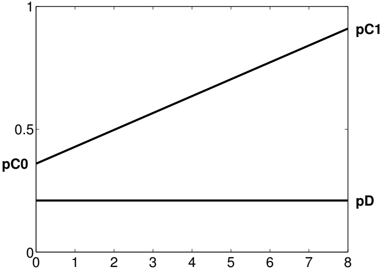
Experiment & Model

Replicator dynamics for the prisoners' dilemma

Jelena Grujić

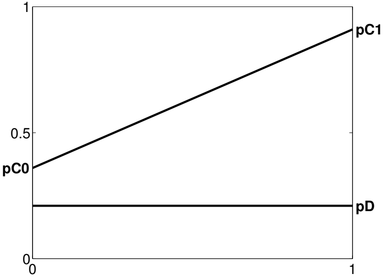
Experiment

- por
- hirt



Model

- por
- hirt



Model

Player's action in the previous round	**C**	**C**	**D**	**D**
Opponent's action in the previous round	**C**	**D**	**C**	**D**
Type 1: mostly defectors	$1-p$	$1-p$	$1-p$	$1-p$
Type 2: mostly cooperators	p	p	p	p
Type 3: conditional cooperators	p_{C1}	p_{C0}	p_D	p_D

- What are the frequencies?
- Replicator dynamics:

$$\frac{\dot{x}_i}{x_i} = [(\mathbf{A}\vec{x})_i - \vec{x}\mathbf{A}\vec{x}]$$

Payoff matrix

- How to calculate **A**?
- If just C and D, p=1

$$A = \begin{bmatrix} P & T \\ S & R \end{bmatrix}$$

- If $p < 1$ what is A_{11}

$$(1-p)(1-p)R + (1-p)pS + p(1-p)T + ppP$$

- For A_{13} we need transition probability matrix

Transition probability matrix

	C C	**C D**	**D C**	**D D**
C C	M_{11}	M_{12}	M_{13}	M_{14}
C D	M_{21}	M_{22}	M_{23}	M_{24}
D C	M_{31}	M_{32}	M_{33}	M_{34}
D D	M_{41}	M_{42}	M_{43}	M_{44}

Table: Transition probability matrix of the Markov chain generated in an iterated PD game between two given players. The first action is that of the focal player and the second action is that of the opponent. For example, M_{31} is the probability that if focal player played D and his opponent played C, they will both play C in the next round.

Transition probability matrix

Matrix for 1–3 interaction:

$$M(1-3) = \begin{pmatrix} (1-p)p_{C1} & (1-p)(1-p_{C1}) & pp_{C1} & p(1-p_{C1}) \\ (1-p)p_D & (1-p)(1-p_D) & pp_D & p(1-p_D) \\ (1-p)p_{C0} & (1-p)(1-p_{C0}) & pp_{C0} & p(1-p_{C0}) \\ (1-p)p_D & (1-p)(1-p_D) & pp_D & p(1-p_D) \end{pmatrix}$$

Matrix for 3–3 interaction

$$M(3\text{–}3) = \begin{pmatrix} p_{C1}^2 & (1-p_{C1})\,p_{C1} & (1-p_{C1})\,p_{C1} & (1-p_{C1})^2 \\ p_{C0}p_D & p_{C0}(1-p_D) & (1-p_{C0})\,p_D & (1-p_{C0})(1-p_D) \\ p_{C0}p_D & (1-p_{C0})\,p_D & p_{C0}(1-p_D) & (1-p_{C0})(1-p_D) \\ p_D^2 & (1-p_D)\,p_D & (1-p_D)\,p_D & (1-p_D)^2 \end{pmatrix}$$

Stationary probability vector

- $M^n, n \to \infty$, all members of one column will tend to converge to the same value

$$\pi = \pi \mathbf{M}$$

$$\pi = (\pi_{CC}, \pi_{CD}, \pi_{DC}, \pi_{DD})$$

$$A_{ij} = R\pi_{\mathrm{CC}} + S\pi_{\mathrm{CD}} + T\pi_{\mathrm{DC}} + P\pi_{\mathrm{DD}}$$

- π need to be normalized

Example A

- $p = 0.83,\ p_D = 0.2,\ p_{C0} = 0.4,\ p_{C1} = 0.8$

$$\begin{pmatrix} 0 & 0.235 & -0.026 \\ -0.048 & 0 & 0.062 \\ -0.007 & 0.143 & 0 \end{pmatrix}$$

- Zeeman game:

$$A = \begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}$$

Back to the replicator equation

$$\dot{x}_i = x_i \left[(\mathbf{A}\vec{x})_i - \vec{x}\mathbf{A}\vec{x} \right]$$

- Rest points:

$$0 = x_i \left[(\mathbf{A}\vec{x})_i - \vec{x}\mathbf{A}\vec{x} \right]$$

$$1 = \sum_i x_i$$

- Solutions:

$$(1,0,0),\ (0,1,0),\ (0,0,1),$$

$$\left(\frac{4}{5}, 0, \frac{1}{5}\right),\ \left(0, \frac{5}{8}, \frac{3}{8}\right),\ \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Stability

Linearization

$$\vec{x} \to \vec{sol} + \vec{\varepsilon}, \; \sum_i \varepsilon_i = 0$$

System:

$$\dot{\varepsilon_1} = \frac{17}{9}\varepsilon_1 + \frac{22}{9}\varepsilon_2$$

$$\dot{\varepsilon_2} = -\frac{19}{9}\varepsilon_1 - \frac{23}{9}\varepsilon_2$$

Solutions of the form:

$$\varepsilon_1 = Ae^{mt}$$

$$\varepsilon_2 = Be^{mt}$$

Stability

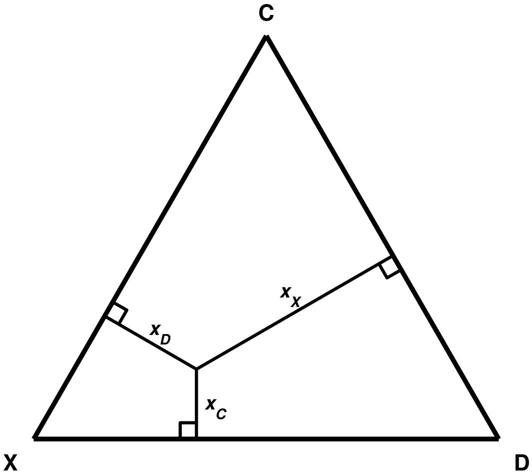
whenever m is a root of the quadratic equation

$$m^2 - \left(\frac{17}{9} - \frac{23}{9}\right) m + \left(\frac{19}{9}\frac{22}{9} - \frac{17}{9}\frac{23}{9}\right) = 0$$

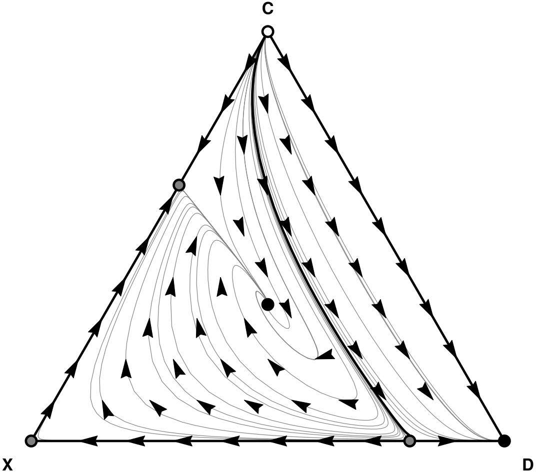
Solutions:

$$m_1 = \frac{1}{3}\left(-1 - i\sqrt{2}\right), \quad m_2 = \frac{1}{3}\left(-1 + i\sqrt{2}\right)$$

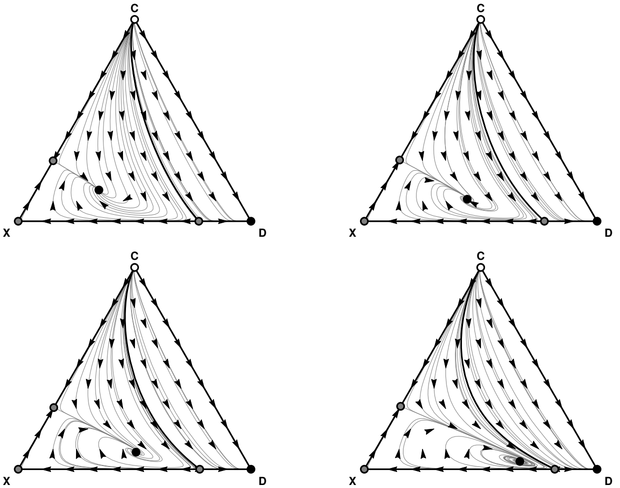
- the roots m_1 and m_2 are conjugate complex $\Rightarrow$ spiral
- real part negative $\Rightarrow$ attractor
- for $(1, 0, 0) m_1 = -3, m_2 = -1 \Rightarrow$ attractor
- others are unstable


C
x_X
x_D
x_C
X
D

Zeeman game

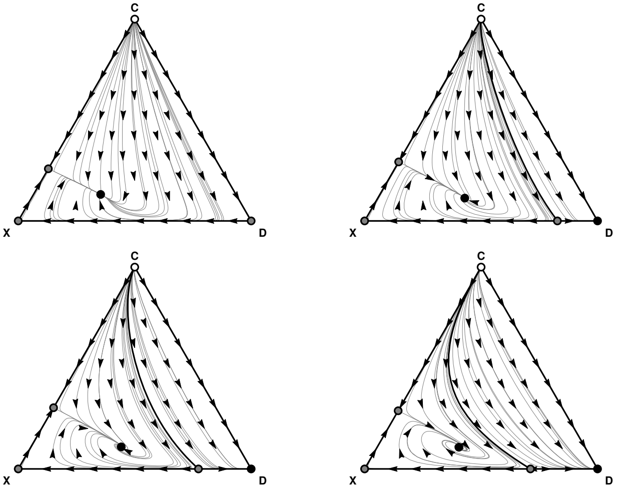


Increasing p



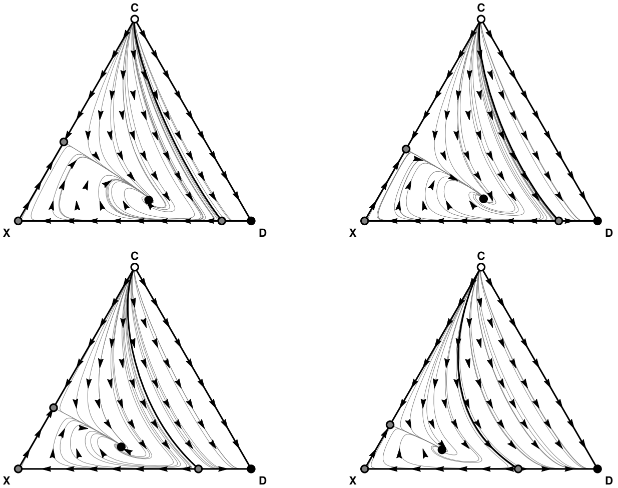
left top $p = 0.8$, right top $p = 0.83$, bottom left $p = 0.85$, bottom right $p = 0.9$

Increasing p_D



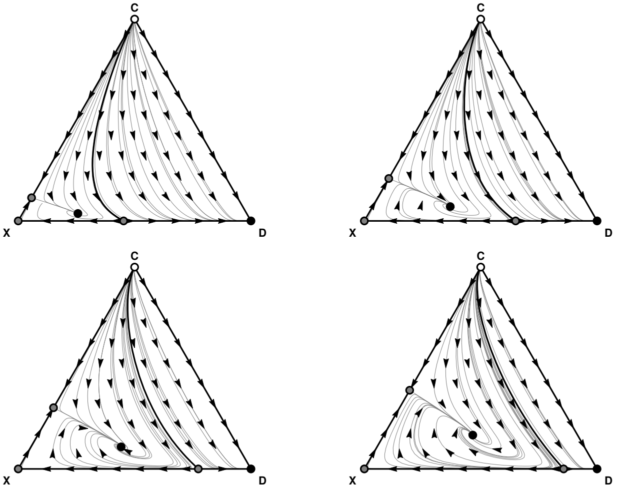
left top $p_D = 0.1$, right top $p_D = 0.15$, bottom left $p_D = 0.2$, bottom right $p_D = 0.3$

Increasing p_{C0}

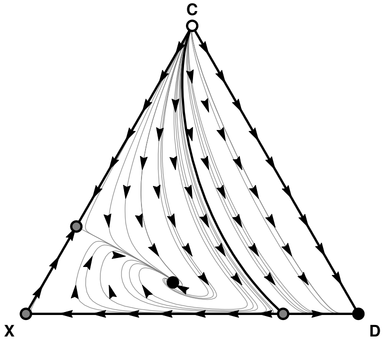


left top $p_{C0} = 0.2$, right top $p_{C0} = 0.3$, bottom left $p_{C0} = 0.4$, bottom right $p_{C0} = 0.5$

Increasing p_{C1}



left top $p_{C1} = 0.7$, right top $p_{C1} = 0.75$, bottom left $p_{C1} = 0.8$, bottom right $p_{C1} = 0.85$



	Defectors	Cooperators	Conditional cooperators
Model	0.39	0.11	0.50
Experiment 1	0.24	0.018	0.74
Experiment 2	0.41	0.053	0.54

Conclusions

- Replicators dynamics for pairwise prisoners' dilemma
- The dynamics exhibits two attractors:
  - one for a population consisting only of defectors,
  - interior point with population frequencies comparable to those observed in the experiment.
  - The former has a much smaller basin of attraction than the latter, which therefore becomes the most probable evolutionary outcome.
- This the first hint that the experiment may be amenable to an evolutionarily explanation.