Multiscattering formalism of Casimir Effect

Pablo Rodriguez-Lopez
parodrilo@gmail.com

UCM

18 - February - 2011

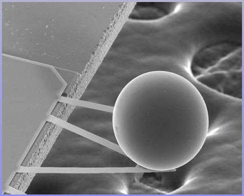
Outline

1. Historical Introduction
2. Multiscattering Formalism of Casimir Energy
3. Casimir energy between non parallel cylinders.
4. Casimir energy between spheres in presence of a plate
5. Pairwise Summation Approximation
6. Casimir energy between topological insulators
7. Conclussions

Colaborators

- Ricardo Brito (UCM)
- Rodrigo Soto (U. Santiago, Chile)
- Thorsten Emig (U Paris-Sud, Paris)
- Sahand Jamal Rahi (MIT)
- Adolfo Grushin (CSIC)
- Alberto Cortijo (CSIC)

- Quantum Vacuum induces an atractive force between uncharged plates because vacuum fluctuations of em field: [Casimir, 1948. Proc. K. Ned. Akad. Wet. 51, 793]
- It has recently been measured with great precision: [Mohideen and Roy, 1998. PRL, 81, 4549]



Casimir Calculation

- Vacuum Energy

$$E = \sum_n \frac{\hbar\omega_n}{2}$$

- Perfect Metal Plates

$$\phi_n(0) = \phi_n(L_x) = 0$$

- Stationary modes of EM field:

$$\phi_n = e^{i\omega_n t} e^{ik_y y} e^{ik_z z} \sin(k_x x) \qquad \omega_n = c\sqrt{k_x^2 + k_y^2 + k_z^2}$$

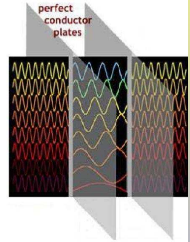
- Having into account the 2 polarizations of EM field:

$$E = 2\frac{\hbar c}{2}\frac{L_y L_z}{(2\pi)^2}\int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \sum_{n=1}^{\infty}\sqrt{\left(\frac{\pi}{L_x}n\right)^2 + k_y^2 + k_z^2} \to \infty$$

Casimir Calculation

Non compensation of vacuum density of energy out and between the plates, there are *more* modes in than out the plates:

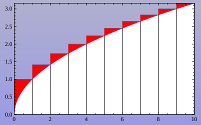
$$\text{Inside:} \quad k_n = \frac{2\pi}{L} n \quad \forall n \in \mathbb{Z}$$
$$\text{Outside:} \quad k_n = \frac{2\pi}{L} n \quad \forall n \in \mathbb{R}$$



Both vacuum energies diverge, but their difference is *finite*:

$$\langle E \rangle_{in} = \frac{\hbar c}{2} \sum_{n \in \mathbb{Z}} k_n \to \infty$$

$$\langle E \rangle_{out} = \frac{\hbar c}{2} \int_{-\infty}^{\infty} k(n) dn \to \infty$$



$$\boxed{\langle E \rangle_{in} - \langle E \rangle_{out} = -\frac{\hbar c \pi^2}{720 L^3}}$$

Partition function with Boundary Conditions

1. Electromagnetic field at a temperature T constrained because N dielectrics. [Emig et. al., PRL, **99**, 170403 (2007).]

$$\mathcal{Z} = \int \mathcal{D}A^{\mu} \prod_{\alpha=1}^{N} \delta\left[C_{\alpha}[A^{\mu}] = 0\right] e^{\frac{-1}{2\hbar c} \int_0^{\hbar c \beta} dx^0 \int d\mathbf{r} A^{\mu} \Box A_{\mu}}$$

2. EM field is bosonic, then
$A^{\mu}(x^0, \mathbf{r}) = A^{\mu}(x^0 + \hbar c \beta, \mathbf{r}) \Rightarrow A^{\mu}(x^0, \mathbf{r}) = \sum_{n \in \mathbb{Z}} A^{\mu}_n(\mathbf{r}) e^{i \kappa_n x^0}$

$$\mathcal{Z} = \prod_{n \in \mathbb{Z}} \mathcal{Z}_n$$

3. Functional Dirac delta:

$$\delta\left[C_{\alpha}[A^{\mu}_n] = 0\right] = \int \mathcal{D}j_{\mu,n,\alpha} e^{\int_{\alpha} d\mathbf{r}_{\alpha} j_{\mu,n,\alpha} A^{\mu}_n}$$

$$\mathcal{Z}_n = \int \mathcal{D}A_{\mu} \prod_{\alpha=1}^{N} \int \mathcal{D}j^{\mu}_{\alpha} e^{-\frac{\beta}{2} \int d\mathbf{r} A^{\mu}\left(\Delta + \kappa_n^2\right) A_{\mu} + \sum_{\alpha=1}^{N} \int_{\alpha} d\mathbf{r}_{\alpha} j^{\mu}_{\alpha} A_{\mu}}$$

Partition function with Boundary Conditions

4. Carrying out the integral over A_μ we obtain:

$$\mathcal{Z}_n = \prod_{\alpha=1}^{N} \int \mathcal{D} j_\alpha^\mu e^{-\frac{1}{2\beta} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \int_\alpha d\mathbf{r}_\alpha \int_\beta d\mathbf{r}_\beta j_\alpha^\mu(\mathbf{r}_\alpha) G_{\mu\nu}(\mathbf{r}_\alpha, \mathbf{r}_\beta, \kappa_n) j_\beta^\nu(\mathbf{r}_\beta)}$$

5. Multipolar expansion of currents $j = \sum_n j_n Q_n$.

$$G_{ij}(\mathbf{r}, \mathbf{r}', \kappa) = \left[\delta_{ij} - \frac{1}{\kappa^2} \nabla_i \nabla'_j\right] \frac{e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\kappa \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \psi_{lm}^{reg}(\kappa\mathbf{r}) \psi_{lm}^{out}(\kappa\mathbf{r}')$$

$$j^\alpha(\mathbf{r}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} Q_{lm}^\alpha \psi_{lm}^{out}(\kappa\mathbf{r}) \qquad Q_{lm}^\alpha = \int d\mathbf{r} j^\alpha(\mathbf{r}) \psi_{lm}^{reg}(\kappa\mathbf{r})$$

$$\mathcal{Z}_n = \prod_{\alpha=1}^{N} \prod_{l,m} \int \mathcal{D} Q_{lm}^\alpha e^{\frac{-1}{2\beta} \sum_{\alpha,l,m} \sum_{\beta,l',m'} Q_{lm}^\alpha \mathbb{M}_{\alpha\beta}^{lm,l'm'} Q_{l'm'}^\beta}$$

Partition function with Boundary Conditions

6. Then the solution is

$$\mathcal{Z} = \frac{1}{\sqrt{|\mathbb{M}|}} \Rightarrow \log(\mathcal{Z}) = -\frac{1}{2}\log|\mathbb{M}|$$

7. With

$$\mathbb{M}_{\alpha\beta} = \delta_{\alpha\beta}\mathbb{T}_{\alpha}^{-1} + (\delta_{\alpha\beta} - 1)\mathbb{U}_{\alpha\beta}$$

- $\mathbb{T}_{\alpha} =$ T matrix of the α-th object.
- $\mathbb{U}^{\alpha\beta} =$ Translation matrix from α-th object to β-th object.

8. Subtract the energy when the objects are at an infinite distance from each other

$$\mathcal{F} = k_B T {\sum_{n=0}^{\infty}}' \left[\log|\mathbb{M}(\kappa_n)| - \log|\mathbb{M}_{\infty}(\kappa_n)|\right]$$

Partition function with Boundary Conditions

9. Then the Helmholtz Free energy is

$$\boxed{\mathcal{F} = k_B T \sum_{n=0}^{\infty}{}' \log |\mathbb{I} - \mathbb{N}(\kappa_n)|}$$

$$\boxed{\lim_{T \to 0} \mathcal{F} = \frac{\hbar c}{2\pi} \int_0^{\infty} d\kappa \log |\mathbb{I} - \mathbb{N}(\kappa)|}$$

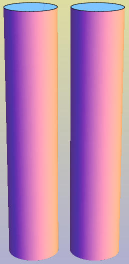
10. With

$$\frac{|\mathbb{M}(\kappa_n)|}{|\mathbb{M}_{\infty}(\kappa_n)|} = |\mathbb{I} - \mathbb{N}(\kappa_n)|$$

11. In particular, for a 2 objects problem

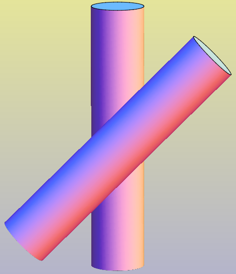
$$\mathbb{N} = \mathbb{T}_1 \mathbb{U}_{12} \mathbb{T}_2 \mathbb{U}_{21}$$

Casimir energy between non parallel cylinders



Parallel Cylinders

$$E \propto L$$



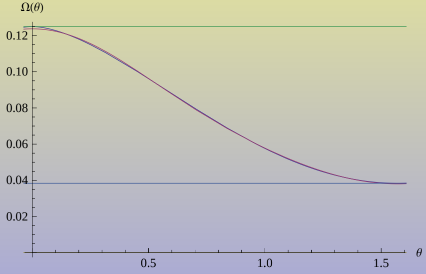
Non Parallel Cylinders

$$E \propto \frac{1}{\sin(\theta)}$$

- Small Radius Approximation

$$E_\theta = -\frac{\hbar c}{8d\sin(\theta)\log^2\left(\frac{R}{2d}\right)}\Omega(\theta)$$

$$E_\parallel = -\frac{\hbar c L}{8\pi d^2\log^2\left(\frac{R}{2d}\right)}$$



- PFA

$$E_\theta = -\frac{\hbar c\pi^3}{720\sin(\theta)}\frac{R}{(d-2R)^2}$$

$$E_\parallel = -\frac{\hbar c L\pi^3}{1920}\sqrt{\frac{R}{(d-2R)^5}}$$

Numerical results

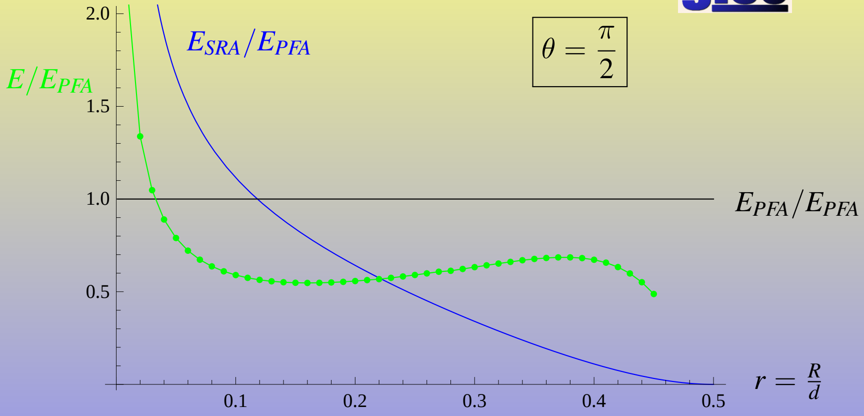
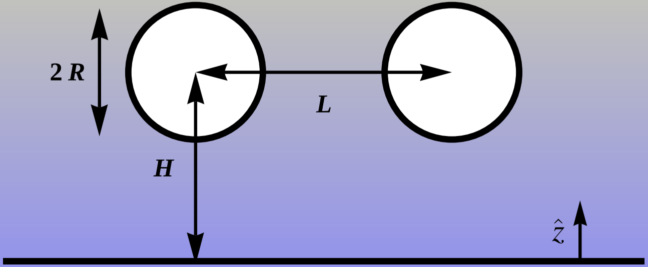


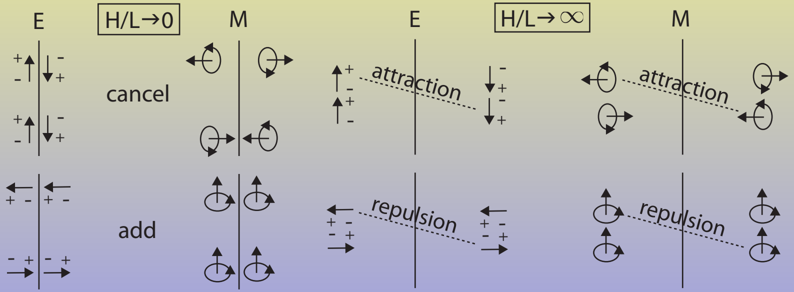
Figura: Numerical Results

Casimir energy between 2 spheres & a plate

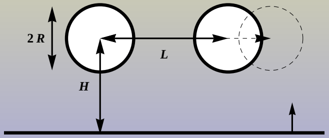
- How is modified the force between 2 bodies when a third enters in the system? [P. Rodriguez-Lopez et. al., Phys. Rev. A **80**, 022519 (2009)]
- Holds Casimir forces a superposition principle?

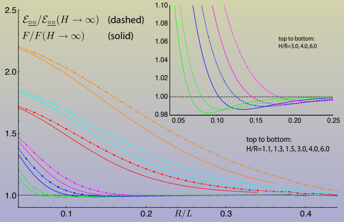


Anisotropic Dipoles

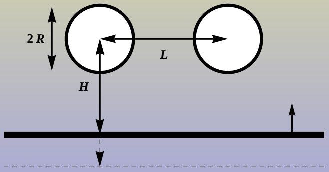


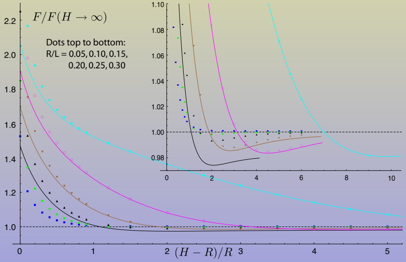
2 perfect metal spheres





2 perfect metal spheres





Pairwise Summation Approximation

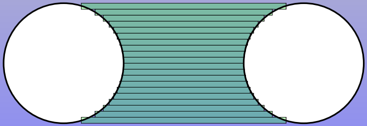
- Approximations to Casimir Interaction
  - Proximity Force Approximation (PFA)
  - Pairwise Summation Approximation (PSA) [Lifshitz, Sov. Phys. Rev. **73**, 360 (1948)] , [Golestanian, PRL, **95**, 230601 (2005)] , [Milton et. al. PRL,. **101**, 160402 (2008)] .
- When PSA is a valid approximation?

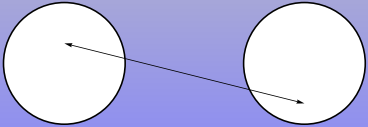
$$\frac{d^2E}{dA} = -\frac{\hbar c \pi^2}{720 R^3}$$

$$E = \int_s dA \frac{d^2E}{dA}$$

$$\frac{d^6E}{dV_1 dV_2} = -\frac{23\hbar c}{4\pi R^7}\alpha_1^E \alpha_2^E$$

$$E = \int_1 dV_1 \int_2 dV_2 \frac{d^6E}{dV_1 dV_2}$$





Our approximation to the problem

- EGJK formula

$$E = \frac{\hbar c}{2\pi} \int_0^\infty d\kappa \log \left| \mathbb{I} - \mathbb{T}^1 \mathbb{U}^{12} \mathbb{T}^2 \mathbb{U}^{21} \right|$$

- Expansion of log-det formula

$$\log |\mathbb{I} - \mathbb{N}| = \mathrm{Tr} \log (\mathbb{I} - \mathbb{N}) = -\sum_{p=1}^{\infty} \frac{1}{p} \mathrm{Tr} (\mathbb{N})^p$$

- Small $\mathbb{N}$ approximation

$$E \approx -\frac{\hbar c}{2\pi} \int_0^\infty d\kappa \mathrm{Tr} (\mathbb{N})$$

Born approximation of $\mathbb{T}$ matrix

- $S = \frac{1}{2}\int_\Omega dx^\mu \left(\epsilon \mathbf{E}^2 + \mu \mathbf{H}^2\right)$
- $S = \frac{1}{2}\int_\Omega dx^\mu \left(\epsilon_0 \mathbf{E}^2 + \mu_0 \mathbf{H}^2\right) + \sum_{i=1}^N \frac{1}{2}\int_{\Omega_i} dx^\mu \left(\left(\epsilon_i - \epsilon_0\right)\mathbf{E}^2 + \left(\mu_i - \mu_0\right)\mathbf{H}^2\right)$

$$\mathbb{V}_i = \begin{pmatrix} V_E & 0 \\ 0 & V_H \end{pmatrix} = \begin{pmatrix} \tilde{\epsilon}_i & 0 \\ 0 & \tilde{\mu}_i \end{pmatrix} \chi_i(\mathbf{r}) \qquad (1)$$

- with
  - $\tilde{\epsilon}_i = (\epsilon_i - \epsilon_0)$
  - $\tilde{\mu}_i = (\mu_i - \mu_0)$
  - $\chi_i(\mathbf{r}) = \begin{cases} 1 \ \forall \mathbf{r} \in \Omega_i \\ 0 \ \forall \mathbf{r} \notin \Omega_i \end{cases}$
- Lippmann - Schwinger equation for the $\mathbb{T}$ matrix and Born approximation

$$\mathbb{T} = \left(\mathbb{I} - \mathbb{V}G_0\right)^{-1}\mathbb{V} = \sum_{n=0}^{\infty}\left(\mathbb{V}G_0\right)^n \mathbb{V} \approx \mathbb{V}$$

- Approximation valid for small $\mathbb{V}_i$, it is, small $\tilde{\epsilon}_i$ and $\tilde{\mu}_i$.

$\mathbb{U}$ matrix as matritial Green function

- $G_0\left(R,\kappa\right) = \frac{e^{-\kappa R}}{4\pi R}$, with $R = |\mathbf{r} - \mathbf{r}'|$.

$$\mathbb{U}_{\alpha\beta} = \begin{pmatrix} G^{EE}_{0ij} & G^{EH}_{0ij} \\ G^{HE}_{0ij} & G^{HH}_{0ij} \end{pmatrix} = \begin{pmatrix} [\kappa^2\delta_{ij} + \nabla_i\nabla'_j] & -\kappa\epsilon_{ijk}\nabla_k \\ \kappa\epsilon_{ijk}\nabla_k & [\kappa^2\delta_{ij} + \nabla_i\nabla'_j] \end{pmatrix} G_0\left(R,\kappa\right)$$

- then

$$G^{EE}_{0ij} = G^{HH}_{0ij} = -\frac{e^{-\kappa R}}{4\pi R^3}\left[\left(3 + 3\kappa R + \kappa^2 R^2\right)\frac{R_i R_j}{R^2} + \left(1 + \kappa R + \kappa^2 R^2\right)\delta_{ij}\right]$$

$$G^{EH}_{0ij} = -G^{HE}_{0ij} = \frac{e^{-\kappa R}}{4\pi R}\frac{\kappa}{R}\left(\kappa + \frac{1}{R}\right)\epsilon_{ijk}R_k$$

PSA energy

- Lowest expansion energy

$$E_{PSA} = -\frac{\hbar c}{2\pi}\int_0^\infty d\kappa \mathrm{Tr}\left(\mathbb{N}\right) = -\frac{\hbar c}{2\pi}\int_0^\infty d\kappa \mathrm{Tr}\left(V_1 G_{012} V_2 G_{021}\right)$$

1. Trace over E-H space

$$E_{PSA} = -\frac{\hbar c}{2\pi}\int_0^\infty d\kappa \mathrm{Tr}\begin{pmatrix} V_1^E G_{012}^{EE} V_2^E G_{021}^{EE} + V_1^E G_{012}^{EH} V_2^H G_{021}^{HE} \\ +V_1^H G_{012}^{HE} V_2^E G_{021}^{EH} + V_1^H G_{012}^{HH} V_2^H G_{021}^{HH} \end{pmatrix}$$

$$E_{PSA} = E_{EE} + E_{EH} + E_{HE} + E_{HH}$$

2. Trace over positions (points of the space)

$$\mathrm{Tr}E^{EE} = \mathrm{Tr}\left(V_1^E(\mathbf{r}_1)G_{012}^{EE}(\mathbf{r}_1-\mathbf{r}_2)V_2^E(\mathbf{r}_2)G_{021}^{EE}(\mathbf{r}_2-\mathbf{r}_1')\right)$$

$$\mathrm{Tr}E^{EE} = \int d\mathbf{r}_1 \int d\mathbf{r}_2 \mathrm{Tr}\left(V_1^E(\mathbf{r}_1)G_{012}^{EE}(\mathbf{r}_1-\mathbf{r}_2)V_2^E(\mathbf{r}_2)G_{021}^{EE}(\mathbf{r}_2-\mathbf{r}_1)\right)$$

$$\mathrm{Tr}E^{EE} = \tilde{\epsilon}_1\tilde{\epsilon}_2 \int_{\Omega_1} d\mathbf{r}_1 \int_{\Omega_2} d\mathbf{r}_2 \mathrm{Tr}\left(G_{012}^{EE}(\mathbf{r}_1-\mathbf{r}_2)G_{021}^{EE}(\mathbf{r}_2-\mathbf{r}_1)\right)$$

PSA energy

3. Trace over space coordinates

$$\text{Tr}\left(G_{012}^{EE}G_{021}^{EE}\right) = \frac{e^{-2\kappa R}}{(4\pi)^2 R^6}\left(2\kappa^4 R^4 + 4\kappa^3 R^3 + 10\kappa^2 R^2 + 12\kappa R + 6\right)$$

$$\text{Tr}\left(G_{012}^{EH}G_{021}^{HE}\right) = -\frac{e^{-2\kappa R}}{(4\pi)^2 R^4}\left(2\kappa^2 R^2 + 4\kappa R + 2\right)$$

- Performing the κ integration, we obtain the PSA energy for 2 objects at zero temperature as

$$E = \frac{-\hbar c}{(4\pi)^3}\left[23\tilde{\epsilon}_1\tilde{\epsilon}_2 - 7\tilde{\epsilon}_1\tilde{\mu}_2 - 7\tilde{\mu}_1\tilde{\epsilon}_2 + 23\tilde{\mu}_1\tilde{\mu}_2\right]\int_1\int_2 \frac{d\mathbf{r}_1 d\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^7}$$

Feinberg and Sucher potential

- PSA energy for 2 objects

$$E = \frac{-\hbar c}{(4\pi)^3}\left[23\tilde{\epsilon}_1\tilde{\epsilon}_2 - 7\tilde{\epsilon}_1\tilde{\mu}_2 - 7\tilde{\mu}_1\tilde{\epsilon}_2 + 23\tilde{\mu}_1\tilde{\mu}_2\right]\int_1\int_2 \frac{d\mathbf{r}_1 d\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^7}$$

- Far Distance Approximation: Characteristic radius of the objects much smaller than the distance between them $R = |\mathbf{r}_1 - \mathbf{r}_2|$, in this case:

$$\int_1\int_2 \frac{d\mathbf{r}_1 d\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^7} \approx \frac{V_1 V_2}{R^7}$$

- Polarizabilities are related with permeabilities in the diluted limit as:

$$\alpha^E = \tilde{\epsilon}\frac{V}{4\pi} \qquad\qquad \alpha^H = \tilde{\mu}\frac{V}{4\pi}$$

- Feinberg and Sucher potential as FDA of PSA [Feinberg and Sucher, PRA. **2**, 2395 (1970)]

$$E = \frac{-\hbar c}{4\pi R^7}\left[23\alpha_1^E\alpha_2^E - 7\alpha_1^E\alpha_2^H - 7\alpha_1^H\alpha_2^E + 23\alpha_1^H\alpha_2^H\right]$$

Superposition of Casimir energies in PSA

- EGJK formula for N bodies

$$E_N = k_B T \sum_{n=0}^{\infty}{}' \log\left(\frac{|\mathbb{M}_N(\kappa_n)|}{|\mathbb{M}_{N,\infty}(\kappa_n)|}\right),$$

- $\mathbb{M}_{\alpha\beta} = \delta_{\alpha\beta}\mathbb{T}_\alpha^{-1} + (\delta_{\alpha\beta} - 1)\mathbb{U}_{\alpha\beta}$
- $\mathbb{M}_{\alpha\beta,\infty} = \delta_{\alpha\beta}\mathbb{T}_\alpha^{-1}$

$$|\mathbb{M}_N| = |\mathbb{M}_{N-1}|\,|\mathbb{T}_N|^{-1}\left|\mathbb{I} - \mathbb{T}_N\mathbb{U}_{\gamma,N-1}\mathbb{M}_{N-1}^{-1}\mathbb{U}_{N-1,\gamma}\right|$$

- $\mathbb{M} = \mathbb{T}^{-1} + \mathbb{U} \Rightarrow \mathbb{M}^{-1} = \mathbb{T}\sum_{n=0}^{\infty}(-\mathbb{U}\mathbb{T})^n \Rightarrow \boxed{\mathbb{M}^{-1} \approx \mathbb{T}}$

$$|\mathbb{M}_N| = |\mathbb{M}_{N-1}|\,|\mathbb{T}_N|^{-1}\prod_{m=1}^{N-1}|\mathbb{I} - \mathbb{N}_{lm}|$$

- Pairwise energies in the diluted limit.

$$E_N = k_B T \sum_{n=0}^{\infty}{}' \sum_{l=2}^{N}\sum_{m=1}^{l-1} \log|\mathbb{I} - \mathbb{N}_{lm}|$$

- Corrections gives non-pairwise contributions.

Repulsion in Casimir effect

- Casimir force between dielectrics is (almost) always attractive.
- Casimir force is dominant in nanoscale, it leads to scition.
- Could we obtain repulsive Casimir forces?
- Topological Insulators [A. G. Grushin, P. Rodriguez-Lopez, and A. Cortijo, ArXiv:1102.0455]

$$\vec{D} = \epsilon\vec{E} + \frac{\alpha}{\pi}\theta\vec{B}$$

$$\vec{H} = \mu^{-1}\vec{B} - \frac{\alpha}{\pi}\theta\vec{E}$$

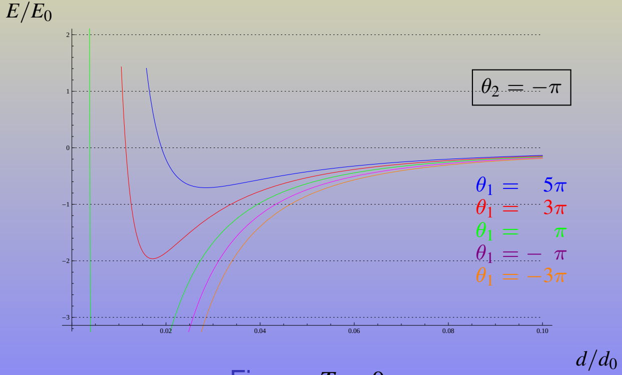


Figura: $T = 0$

Repulsion in Casimir effect

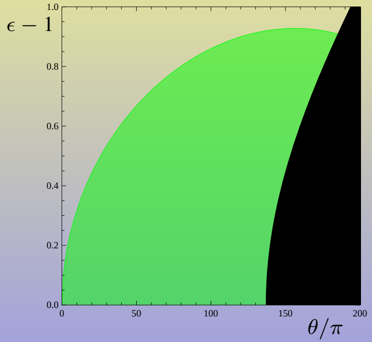


Figura: $T \gg 1$

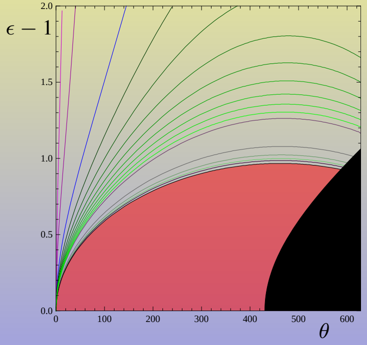


Figura: $T = 0$

Conclussions

- Multiscattering formalism for Casimir effect
- Casimir energy between non parallel cylinders.
- Justification of PSA energy in the diluted limit.
- Superposition behaviour of Casimir energy in the diluted limit.
- Repulsion and Equilibrium distances with Topological Insulators.

THANKS FOR YOUR ATTENTION