Intrinsic fluctuations on Fisher fronts

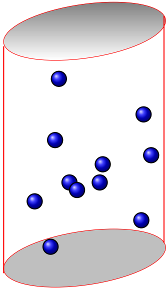
Svetozar Nesic

Departamento de Matemáticas, Universidad Carlos III de Madrid

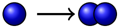
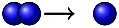
February 8, 2012

GISC workshop, Universidad Carlos III de Madrid

Fisher waves in a Reaction-Diffusion process

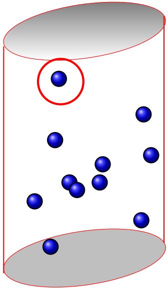


Kinetic reaction:

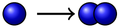
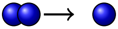
- Creation $A \to 2A$ 
  happens at the rate k
- Annihilation $2A \to A$ 
  happens at the rate λ
- ρ is the concentration of particles

R. A. Fisher, Ann. Eugenics 7, 355 (1937), A. Kolmogorov, I. Petrovsky, N. Piscounov, Moscou Univ. Bull. Math. A 1, 1 (1937)

Fisher waves in a Reaction-Diffusion process

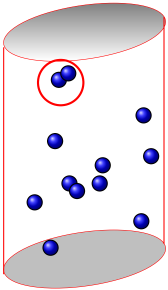


Kinetic reaction:

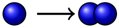
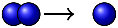
- Creation $A \to 2A$ 
  happens at the rate k
- Annihilation $2A \to A$ 
  happens at the rate λ
- ρ is the concentration of particles

Fisher-Kolmogorov equation

Fisher waves in a Reaction-Diffusion process



Kinetic reaction:

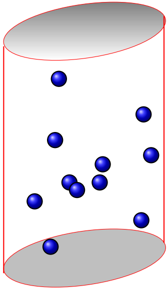
- Creation $A \to 2A$ 
  happens at the rate k
- Annihilation $2A \to A$ 
  happens at the rate λ
- ρ is the concentration of particles

Fisher-Kolmogorov equation

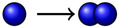
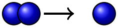
$$\frac{\partial \rho(\vec{r},t)}{\partial t} = \qquad k\rho(\vec{r},t)$$

R. A. Fisher, Ann. Eugenics 7, 355 (1937), A. Kolmogorov, I. Petrovsky, N. Piscounov, Moscou Univ. Bull. Math. A 1, 1 (1937)

Fisher waves in a Reaction-Diffusion process



Kinetic reaction:

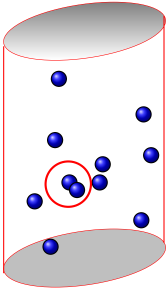
- Creation $A \to 2A$ 
  happens at the rate k
- Annihilation $2A \to A$ 
  happens at the rate λ
- ρ is the concentration of particles

Fisher-Kolmogorov equation

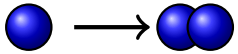
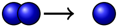
$$\frac{\partial \rho(\vec{r},t)}{\partial t} = \qquad k\rho(\vec{r},t)$$

R. A. Fisher, Ann. Eugenics 7, 355 (1937), A. Kolmogorov, I. Petrovsky, N. Piscounov, Moscou Univ. Bull. Math. A 1, 1 (1937)

Fisher waves in a Reaction-Diffusion process



Kinetic reaction:

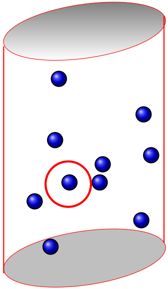
- Creation $A \rightarrow 2A$  happens at the rate k
- Annihilation $2A \rightarrow A$  happens at the rate λ
- ρ is the concentration of particles

Fisher-Kolmogorov equation

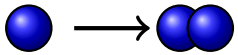
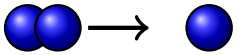
$$\frac{\partial \rho(\vec{r},t)}{\partial t} = \qquad k\rho(\vec{r},t)$$

R. A. Fisher, Ann. Eugenics 7, 355 (1937), A. Kolmogorov, I. Petrovsky, N. Piscounov, Moscou Univ. Bull. Math. A 1, 1 (1937)

Fisher waves in a Reaction-Diffusion process



Kinetic reaction:

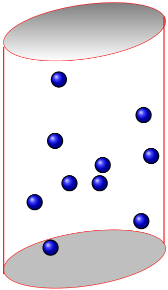
- Creation $A \to 2A$ 
  happens at the rate k
- Annihilation $2A \to A$ 
  happens at the rate λ
- ρ is the concentration of particles

Fisher-Kolmogorov equation

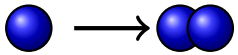
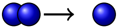
$$\frac{\partial \rho(\vec{r},t)}{\partial t} = k\rho(\vec{r},t) - \lambda\rho^2(\vec{r},t)$$

R. A. Fisher, Ann. Eugenics 7, 355 (1937), A. Kolmogorov, I. Petrovsky, N. Piscounov, Moscou Univ. Bull. Math. A 1, 1 (1937)

Fisher waves in a Reaction-Diffusion process



Kinetic reaction:

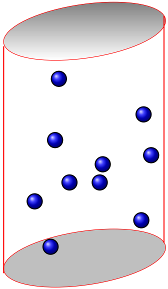
- Creation $A \to 2A$ 
  happens at the rate k
- Annihilation $2A \to A$ 
  happens at the rate λ
- ρ is the concentration of particles

Fisher-Kolmogorov equation

$$\frac{\partial \rho(\vec{r},t)}{\partial t} = D \triangle \rho(\vec{r},t) + k\rho(\vec{r},t) - \lambda \rho^2(\vec{r},t)$$

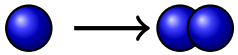
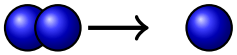
R. A. Fisher, Ann. Eugenics 7, 355 (1937), A. Kolmogorov, I. Petrovsky, N. Piscounov, Moscou Univ. Bull. Math. A 1, 1 (1937)

Fisher waves in a Reaction-Diffusion process



N - number of particles

Kinetic reaction:

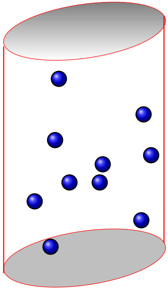
- Creation $A \to 2A$ 
  happens at the rate k
- Annihilation $2A \to A$ 
  happens at the rate λ
- ρ is the concentration of particles

Fisher-Kolmogorov equation (what if ρ is not a continuous function)

$$\frac{\partial \rho(\vec{r},t)}{\partial t} = D\triangle\rho(\vec{r},t) + k\rho(\vec{r},t) - \lambda\rho^2(\vec{r},t)$$

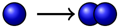
R. A. Fisher, Ann. Eugenics 7, 355 (1937), A. Kolmogorov, I. Petrovsky, N. Piscounov, Moscou Univ. Bull. Math. A 1, 1 (1937)

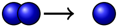
Fisher waves in a Reaction-Diffusion process



N - number of particles

Kinetic reaction:

- Creation $A \rightarrow 2A$ 
  happens at the rate k

- Annihilation $2A \rightarrow A$ 
  happens at the rate λ

- ρ is the concentration of particles

Fisher-Kolmogorov-Petrovsky-Piscunov equation

$$\frac{\partial \rho(\vec{r},t)}{\partial t} = D \triangle \rho(\vec{r},t) + k\rho(\vec{r},t) - \lambda \rho^2(\vec{r},t) + \sqrt{\rho(\vec{r},t)/N}\,\eta(\vec{r},t)$$

R. A. Fisher, Ann. Eugenics 7, 355 (1937), A. Kolmogorov, I. Petrovsky, N. Piscounov, Moscou Univ. Bull. Math. A 1, 1 (1937)

Master equation approximated by a diffusion process

$$\frac{\partial P_n(t)}{\partial t} = \frac{k}{V}(n-1)P_{n-1} - \frac{k}{V}nP_n - \frac{\lambda}{V}\frac{n(n-1)}{2}P_n + \frac{\lambda}{V}\frac{n(n+1)}{2}P_{n+1}$$

Van Kampen's system size expansion $\rightarrow$ Fokker-Planck equation

$$\frac{\partial P(\rho,t)}{\partial t} = -\frac{(k_1-k_2)N}{V}\frac{\partial}{\partial\rho}\rho(1-\rho)P(\rho,t) + \frac{1}{2}\frac{k_1+k_2}{V}\frac{\partial^2}{\partial\rho^2}\rho(1-\rho)P(\rho,t)$$

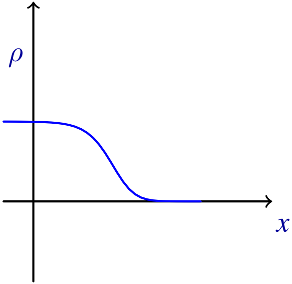
where $\gamma = \frac{(k_1-k_2)N}{V}$, in the thermodynamic limit $V \sim N$ and $\sigma^2 = \frac{k_1+k_2}{V} \simeq \frac{1}{N}$ gives the Langevin equation

$$\frac{\partial\rho(t)}{\partial t} = \gamma\rho(1-\rho) + \frac{1}{\sqrt{N}}\sqrt{\rho-\rho^2}\,\eta$$

C. R. Doering, P. Smereka, Physica A, 325 (2003)

Logistic equation (modeling population growth)

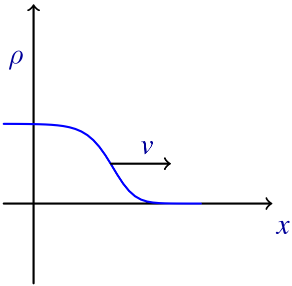
$$\frac{\partial \rho}{\partial t} = \rho(1 - \rho)$$



J.D. Murray, Mathematical Biology: I. An Introduction, Springer, 1989

Fisher-Kolmogorov model:

$$\frac{\partial \rho}{\partial t} = \rho(1-\rho) + D\Delta\rho$$



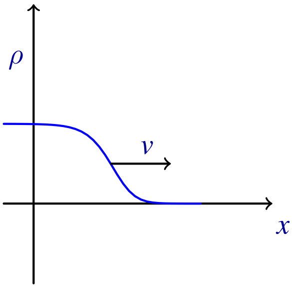
Velocity of the front:

$$v \quad = 2\sqrt{D}$$

J.D. Murray, Mathematical Biology: I. An Introduction, Springer, 1989

Fisher-Kolmogorov-Petrovsky-Piscunov model:

$$\frac{\partial \rho}{\partial t} = \rho(1-\rho) + D\Delta\rho + $$
$$+\sqrt{\rho/N}\eta$$



Velocity of the front - Brunet-Derrida formula:

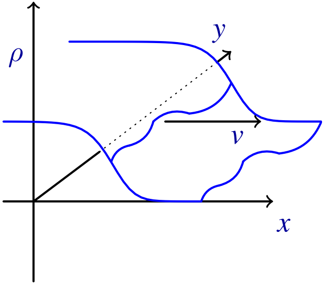
$$v_N = 2\sqrt{D} - \frac{C}{\log^2 N}$$

E. Brunet and B. Derrida, Phys. Rev. E 56, 2597 (1997)

Fisher-Kolmogorov-Petrovsky-Piscunov model in 2d:

$$\frac{\partial\rho}{\partial t} = \rho(1-\rho) + D\Delta\rho + $$

$$+\sqrt{\rho/N}\eta$$

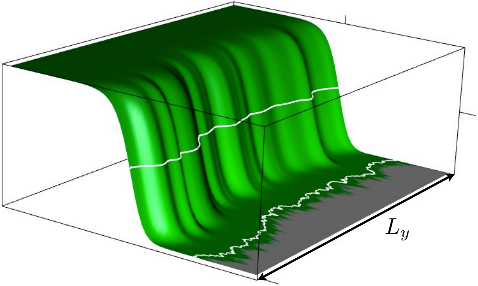


What is the dynamics of two dimensional fronts?

How does the surface interface scale?

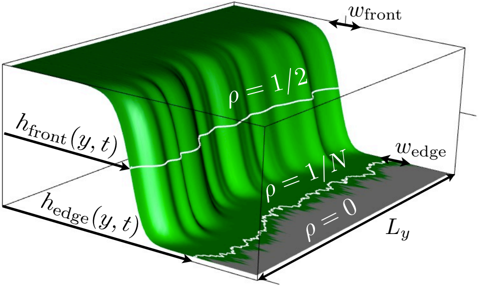
A. L. Barabasi, H. E. Stanley, Fractal Concepts in Surface Growth, CUP, 1995

Fisher-Kolmogorov-Petrovsky-Piscunov model in 2d:



$$w^2_{\text{front}}(t) = \frac{1}{L_y}\int_0^{L_y} \left(h_{\text{front}} - \bar{h}\right)^2 dy$$

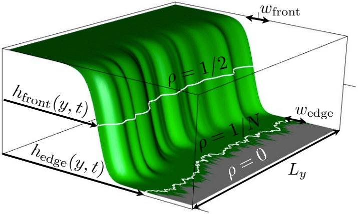
Fisher-Kolmogorov-Petrovsky-Piscunov model in 2d:



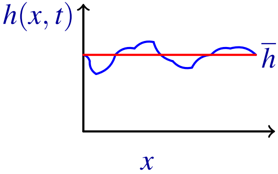
$$w^2_{\text{front}}(t) = \frac{1}{L_y}\int_0^{L_y} \left(h_{\text{front}} - \bar{h}\right)^2 dy$$

Kinetic roughening

2D Fisher front



Intrinsic noise roughens the surface of the front.



Roughness function

$$W(L,t) = \sqrt{\frac{1}{L}\sum_{i=1}^{L}(h(i,t) - \bar{h}(t))^2}.$$

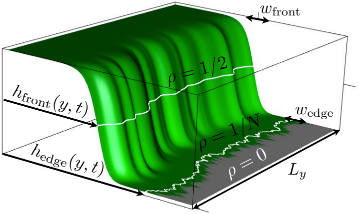
Structure factor

$$S(q) = \langle h_q h_{-q} \rangle$$

A. L. Barabasi, H. E. Stanley, Fractal Concepts in Surface Growth, CUP, 1995

Kinetic roughening

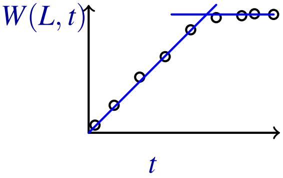
2D Fisher front



Family-Vicsek scaling Ansatz

$$W(L,t) = L^{\alpha} f\left(\frac{t}{L^z}\right)$$

$$S(q, t \to \infty) \sim \frac{1}{q^{2\alpha+d}}$$



- Growth $W(L,t) \sim t^{\beta}$
- Saturation $W(L,t) \sim L^{\alpha}$
- Surface saturates at $t_x \sim L^z$

Universality class $\iff$ (α, β, z)

$$\alpha = z\beta$$

A. L. Barabasi, H. E. Stanley, Fractal Concepts in Surface Growth, CUP, 1995

Two different results using particle model for $A \to 2A$ and $2A \to A$ process

Conjecture:

In general, 2 dimensional Fisher waves belong to the KPZ universality class in 2 dimensions, even though the front is described by a 1 dimensional interface!

G. Tripathy and W van Saarloos, Phys. Rev. Lett. 85, 3556 (2000)

2 dimensional Fisher waves belong to the KPZ universality class in the 1 dimension.

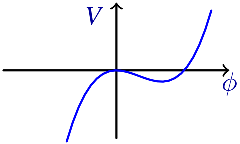
E. Moro, Phys. Rev. Lett. 87, 238303 (2001)

Conjecture:

Stochastic equation for pulled fronts:

$$\frac{\partial \phi}{\partial t} = D\triangle\phi + f(\phi) + g(\phi)\eta$$

where $V'(\phi) = f(\phi) = \phi - \phi^3$



and $g(\phi) = \sigma(\phi - \phi^3)^\alpha$, belong to the KPZ universality class of the same dimensionality!

Leading edge transformation $\phi \sim 0$

$\phi = e^{-\lambda\xi}\psi$ and $\alpha = 1$ give

$$\frac{\partial \psi}{\partial t} = D\triangle\psi + \psi\eta$$

Cole-Hopf transformation $\psi = e^h$ gives KPZ equation

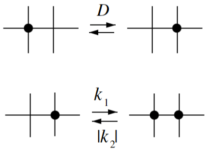
$$\frac{\partial h}{\partial t} = D\triangle h + D(\nabla h)^2 + \eta$$

Only works when $\alpha = 1$.

G. Tripathy and W van Saarloos, Phys. Rev. Lett. 86, 5215 (2001)
U. Ebert and W van Saarloos, Phys. Rev. Lett. 80, 1650 (1998)

Particle model:

When N is really large, simulations are costly!



G. Tripathy, W. van Saarloos, Phys. Rev. Lett. 85, 3556 (2000)
C. R. Doering, M. A. Burschka, W. Horsthemke, J. Stat. Phys. 65, 953 (1991)

VS

Splitting step scheme:

$$\frac{\partial \rho}{\partial t} = \sigma \sqrt{\rho}\, \eta \quad \Longleftrightarrow \quad \lambda \rho = \mathrm{Gamma}[\mathrm{Poisson}(\lambda \rho_0)]$$

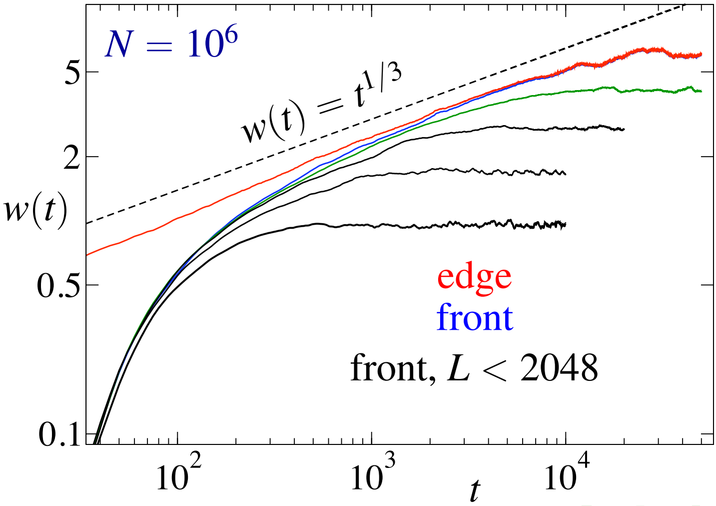
$\rho \to \rho^*$

$$\frac{\partial \rho^*(x,t)}{\partial t} = D \frac{\partial^2 \rho^*(x,t)}{\partial x^2} + \rho^*(x,t) - \rho^{*2}(x,t)$$

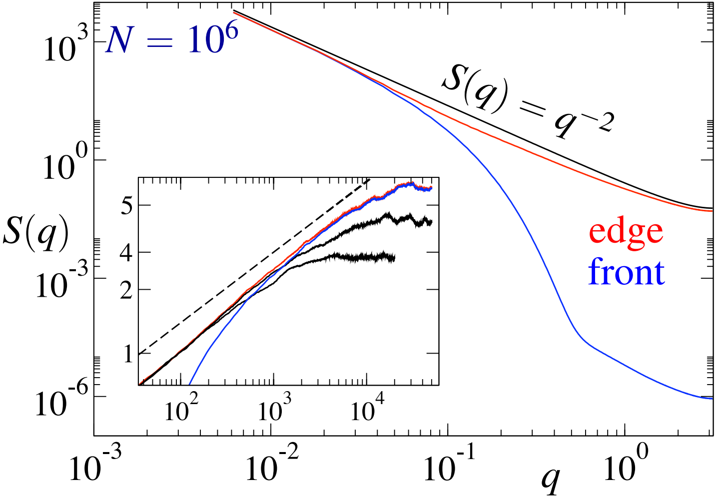
$\rho^* \to \rho_0$

I. Dornic, H. Chate, M. A. Muñoz, Phys. Rev. Lett. 94, 100601 (2005)
E. Moro, Phys. Rev. E, 70, 045102 (2004)

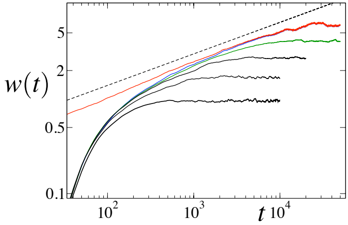
Roughness
of the front and the edge line



Structure factor
of the front and the edge line

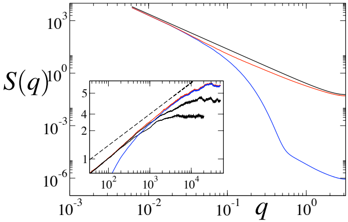


Roughness



$\beta = \frac{1}{3}$

Structure factor



$\alpha = \frac{1}{2}$

Both macroscopic front line and microscopic edge line belong to $1d$ KPZ universality class!

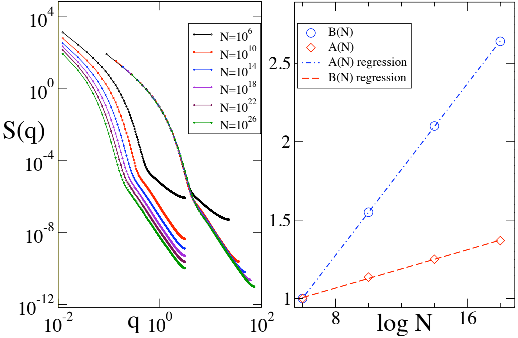
Structure factor scaling:

In KPZ region we expect scaling:

$$\frac{S_N(q/B(N))}{A(N)} = f(q)$$

we scale curves to the curve for which $N = 10^6$

$$A(\sigma) = a + a_1 \log N$$
$$B(\sigma) = b + b_1 \log N$$

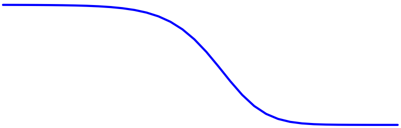


When $q \to 0$ we have $S(q) = \frac{T_{eff}}{q^2}$ we get that effective temperature scales as:

$$T'_{eff} = \frac{TB^2}{A} \simeq \frac{T_{eff}}{\log N}$$

One more 1d result

Traveling wave solution:



- No physical significance of $\rho < \frac{1}{N}$
- δ - distance ahead of the edge
- The position of the front changes

$$X_{t+\delta t} = \begin{cases} X_t + v_{cutoff}\,\Delta t + R(\delta) & \text{with prob. } P(\delta) \\ X_t + v_{cutoff}\,\Delta t & \text{with prob. } 1 - P(\delta) \end{cases}$$

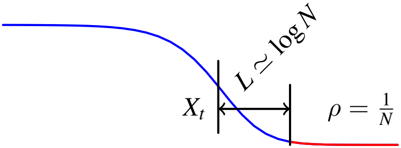
where $P(\delta)$ is the probability of a relevant fluctuation during the time Δt

The effect of noise is so weak that, most of time, it can be ignored and the cutoff theory describes accurately the evolution of the front!

E. Brunet, B. Derrida, A. H. Mueller, and S. Munier, Phys. Rev. E 73, 056126 (2006)

One more 1d result

Traveling wave solution:



- No physical significance of $\rho < \frac{1}{N}$
- δ - distance ahead of the edge
- The position of the front changes

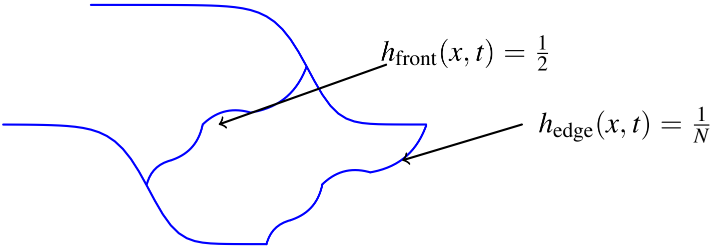
$$X_{t+\delta t} = \begin{cases} X_t + v_{cutoff}\,\Delta t + R(\delta) & \text{with prob. } P(\delta) \\ X_t + v_{cutoff}\,\Delta t & \text{with prob. } 1 - P(\delta) \end{cases}$$

where $P(\delta)$ is the probability of a relevant fluctuation during the time Δt

The effect of noise is so weak that, most of time, it can be ignored and the cutoff theory describes accurately the evolution of the front!

E. Brunet, B. Derrida, A. H. Mueller, and S. Munier, Phys. Rev. E 73, 056126 (2006)

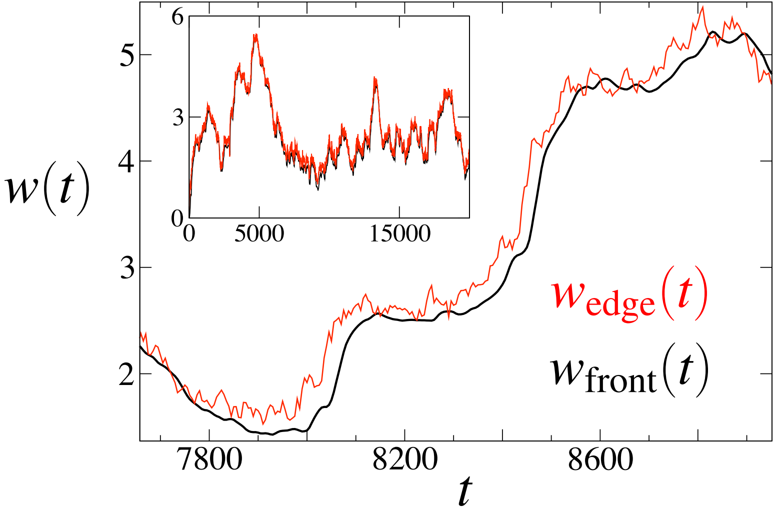
Correlation between the front and the tip



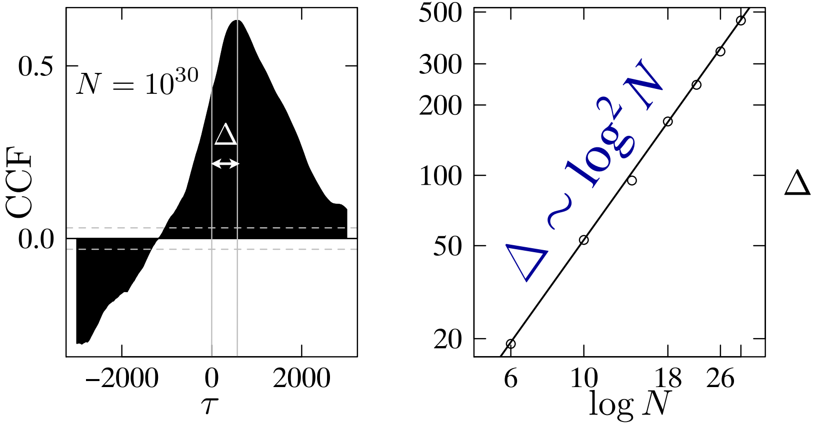
$$W_{\text{front}}(t) \sim a W_{\text{edge}}(t-\Delta) + \text{noise}$$

Simulation of a Fisher front - one realization!

Cross-Correlation between $W_{\mathbf{front}}(t)$ and $W_{\mathbf{edge}}(t)$



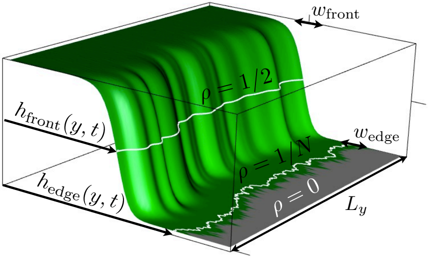
Cross-Correlation between $W_{\text{front}}(t)$ and $W_{\text{edge}}(t)$



Cross-Correlation between $W_{\mathbf{front}}(t)$ and $W_{\mathbf{edge}}(t)$

Strong fluctuations at the tip will happen at the front after

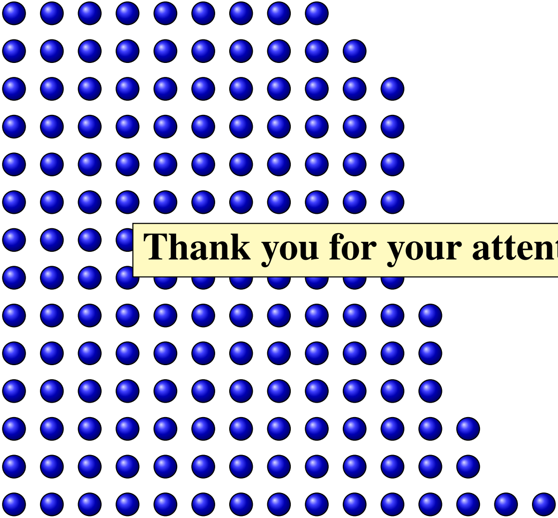
$$\Delta \sim \ln^2 N$$



Microscopic tip fluctuations drive the morphology of macroscopic Fisher fronts

Conclusions:

- One dimensional equipotential lines of stochastic Fisher fronts belong to the one dimensional KPZ universality class contrary to the conjecture!
- Even for a large number of particles the front still feels the noise!
- Fluctuations propagate from the edge to the front providing the same universality class for all equipotential lines.



Thank you for your attention