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Lump solitons in a higher-order nonlinear equation in 2 + 1 dimensions

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We propose and examine an integrable system of nonlinear equations that generalizes the nonlinear Schrödinger equation to 2 + 1 dimensions. This integrable system of equations is a promising starting point to elaborate more accurate models in nonlinear optics and molecular systems within the continuum limit. The Lax pair for the system is derived after applying the singular manifold method. We also present an iterative procedure to construct the solutions from a seed solution. Solutions with one-, two-, and three-lump solitons are thoroughly discussed.

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I. INTRODUCTION

The cubic nonlinear Schrödinger equation (NLSE) with additional high-order dispersion terms emerges very often in the theoretical description of a number of physical problems in molecular systems, nonlinear optics, and fluid dynamics, to name a few. For instance, the propagation of energy released during adenoside triphosphate hydrolysis, through amide-I vibrations along the hydrogen bonding spine of the α-helical proteins, is described by a set of equations which, for dipole-dipole interaction, in the lower order of the continuum approximation is governed by the NLSE. Some years ago, in an attempt to extend the Davydov model for the energy transfer of local vibrational modes in α-helical proteins [1,2], Daniel and Deepmala considered the effects of higher-order molecular excitations [3] that introduce quadrupole-quadrupole coefficients. The result of such a generalization is a NLSE that includes a fourth-order dispersion term, the so-called Lakshmanan-Porsezian-Daniel equation:

$$i\psi_t + \tfrac{1}{2}\psi_{xx} + |\psi|^2\psi + \gamma\big(\psi_{xxxx} + 6\psi_x^2\psi^* + 4|\psi_x|^2\psi + 8\psi_{xx}|\psi|^2 + 2\psi_{xx}^*\psi^2 + 6\psi|\psi|^4\big) = 0, \tag{1}$$

whose integrability and soliton solutions were studied in Refs. [3] and [4]. More recently, Ankiewicz *et al.* proposed a further generalization of the NLSE, adding a third-order dispersion term [5]. The integrability of this extended NLSE for some values of the parameters of the equation was confirmed in Ref. [6], where Lax operators were presented. This integrable version appears in the mentioned references as

$$i\psi_t + \tfrac{1}{2}\psi_{xx} + |\psi|^2\psi + \gamma\big(\psi_{xxxx} + 6\psi_x^2\psi^* + 4|\psi_x|^2\psi + 8\psi_{xx}|\psi|^2 + 2\psi_{xx}^*\psi^2 + 6\psi|\psi|^4\big) = i\alpha(\psi_{xxx} + 6\psi_x|\psi|^2). \tag{2}$$

This equation contains many integrable particular cases such as the standard NLSE ($\alpha = \gamma = 0$) [7], the Hirota equation ($\gamma = 0$) [8], and the Lakshmanan-Porsezian-Daniel equation ($\alpha = 0$) [4]. Soliton solutions and rogue wave for this equation can be found in Refs. [6] and [7] as well as in Refs. [9] and [10]. The relevance of third-order dispersion terms in the context of the self-induced Raman effect have been pointed out by Hesthaven *et al.* [11]. Moreover, rogue waves in optical fibers can be mathematically described by the NLSE equation and its extensions that take into account third-order dispersion [12].

Models discussed so far are defined in 1 + 1 dimensions as they are aimed at describing the dynamics of the excitations in a single strand of the protein. These models need to include more degrees of freedom and more spatial dimensions to cope with the complex helical geometry of the proteins. To this aim, there exist different generalizations of the NLSE to 2 + 1 dimensions. In particular, we can consider the following system proposed by Calogero in Ref. [13], and then discussed by Zakharov [14], which trivially reduces to the NLSE on the line $x = y$.

$$iu_t + u_{xy} + 2um_y = 0, \quad -iw_t + w_{xy} + 2wm_y = 0, \quad m_x + uw = 0, \tag{3}$$

where $w = u^*$. This equation has been studied by different authors. A derivation of the Lax pair and Darboux transformations by means of the singular manifold method appears in Ref. [15]. The same method was applied in Ref. [16] to derive rational solitons (lumps) of a different generalization of the NLSE to 2 + 1 dimensions. Notice that the second derivative includes crossed terms u_{xy} instead of some combination of u_{xx} and u_{yy} as appears in many genaralizations of NLS. They could be easily recovered through the change of variables $x = \hat{x} + \hat{y}$ and $y = \hat{x} - \hat{y}$ that yields $u_{xy} = u_{\hat{x}\hat{x}} - u_{\hat{y}\hat{y}}$.

Rogue waves in 1 + 1 dimensions [17] as well as lumps in 2 + 1 dimensions [16] are rational solutions with nontrivial behavior. This suggests that rogue waves can appear as a reduction of variables in the lump solutions. As it is well known, lumps are solutions whose meromorphic structure guarantees their stability [18]. This is the main motivation to propose a modified NLSE in 2 + 1 dimensions similarly to the generalization considered in Ref. [15] but including also third- and fourth-order dispersion terms, as in the case of Eq. (2), to be a good candidate for the continuum limit of different discrete models that have been proposed to describe the dynamics of α-helical proteins. The proposed generalization of the set (3) can be cast as a system of equations in the following form:

$$iu_t + u_{xy} + 2um_y + i\alpha(u_{xxx} - 6uwu_x) + \gamma\big(u_{xxxx} - 8uwu_{xx} - 2u^2w_{xx} - 4uu_xw_x - 6wu_x^2 + 6u^3w^2\big) = 0, \tag{4a}$$

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$$-iw_t + w_{xy} + 2wm_y - i\alpha(w_{xxx} - 6uww_x) + \gamma\big(w_{xxxx} - 8uww_{xx} - 2w^2u_{xx} - 4wu_xw_x - 6uw_x^2 + 6u^2w^3\big) = 0, \tag{4b}$$

$$m_x + uw = 0. \tag{4c}$$

Notice that (4b) is the complex conjugate of (4a) and m is a real field.

It can be easily checked that (4) has the Painlevé property for arbitrary α and γ. Therefore, it can be regarded as an integrable system of equations, which means that it should have a Lax representation. A powerful method to derive the Lax pair is the singular manifold method [19], which has already been applied to the above-mentioned generalizations of the NLSE to $2+1$ dimensions [15,16].

The rest of the paper is organized as follows. In Sec. II we apply the singular manifold method to (4) in order to derive the Lax pair for this system. The application of the singular manifold method to the Lax pair itself provides a Darboux transformation as well as an iterative method of building up solutions [15] that allows us to obtain the Hirota τ function for each iteration. Darboux transformation and τ function will be the aim of Secs. III and IV. Section V will be devoted to obtain solutions of (4) by using a trivial seed solution. Section VI concludes the paper.

It is compulsory to say that we have not exhausted all possible solutions of (4). We are restricted to lump solutions for the reasons above mentioned. Nevertheless, many more solutions can be obtained whose behavior could be dependent on the values of α and γ. The study of these solutions can be the objective of future research.

II. SINGULAR MANIFOLD METHOD

As is well known [19], the Painlevé property for (4) means that the fields u, w, and m can be expanded through a generalized Laurent expansion of the form

$$u = \sum_{j=0}^{\infty} a_j(x,y,t)[\phi(x,y,t)]^{j-a},$$

$$w = \sum_{j=0}^{\infty} b_j(x,y,t)[\phi(x,y,t)]^{j-b},$$

$$m = \sum_{j=0}^{\infty} m_j(x,y,t)[\phi(x,y,t)]^{j-c}, \tag{5}$$

where $\phi(x,y,t)$ is an arbitrary function. After substitution of (5) into (4), a leading-order analysis leads to $a = b = c = -1$. This means that all solutions are single valued around the singularity manifold $\phi(x,y,t) = 0$. In addition, we get

$$a_0 b_0 = \phi_x^2, \tag{6a}$$

$$c_0 = \phi_x. \tag{6b}$$

It is worth mentioning that (6a) implies that there exists a resonance in $j = 0$, which means that a_0 and b_0 cannot be independently fixed.

A. Truncated expansion

The singular manifold method implies the truncation of the Painlevé expansion (5) at the constant level $j = 1$. Let $\phi_1(x,y,t)$ be the singular manifold for this truncation. In this case, (6a) allows us to introduce a function $g_1(x,y,t)$ such that

$$a_0 = g_1\phi_{1,x}, \quad b_0 = \frac{\phi_{1,x}}{g_1}, \quad c_0 = \phi_{1,x}, \tag{7}$$

and the truncation of (5) can be written as

$$u^{[1]} = u^{[0]} + \frac{g_1\phi_{1,x}}{\phi_1},$$

$$w^{[1]} = w^{[0]} + \frac{\phi_{1,x}}{g_1\phi_1},$$

$$m^{[1]} = m^{[0]} + \frac{\phi_{1,x}}{\phi_1}, \tag{8}$$

where we have defined $u^{[0]} = a_1$, $w^{[0]} = b_1$, and $m^{[0]} = c_1$ because, after substitution of (8) into (4), it is easy to see that the terms of power 0 in ϕ_1 should also be solutions of the equation. Solutions obtained through this truncated expansion in the *singular manifold* ϕ_1 have been denoted as $(u^{[1]},w^{[1]},m^{[1]})$ accordingly. Actually, the truncation (8) can be understood as an auto-Bäcklund transformation that relates two different solutions of (4). Furthermore, truncation (8) also implies an iterative method of construction of solutions where the superindex [0] denotes a seed solution and [1] the iterated one.

B. Singular manifold equations

Substitution of (8) in (4) leads to three polynomials in powers of ϕ_1. By imposing that each coefficient vanishes, we obtain a set of equations: *the singular manifold equations* that relates the singular manifold ϕ_1 with the seed solution $(u^{[0]},w^{[0]},m^{[0]})$. The process of obtaining these equations requires some tedious but straightforward calculation that we have performed with the aid of the symbolic calculus package MAPLE. The result can be summarized as follows:

$$u^{[0]} = \frac{g_1}{2}(-v_1 - h_{1,x} - 2i\lambda_1),$$

$$w^{[0]} = \frac{1}{2g_1}(-v_1 + h_{1,x} + 2i\lambda_1), \tag{9a}$$

in addition to

$$m_y^{[0]} = -\frac{v_{1,y}}{2} - \frac{1}{2}(ih_{1,t} + h_{1,x}h_{1,y}) - \frac{i\alpha}{2}\left\{h_{1,xxx} + 3\left[s_1 + \frac{(h_{1,x}+2i\lambda_1)^2}{2}\right]h_{1,x} + h_{1,x}^3\right\} - \gamma\left\{\frac{s_{1,x,x}}{2} + \frac{3}{4}\left[s_1 + \frac{(h_{1,x}+2i\lambda_1)^2}{2}\right]^2 + 3s_1h_{1,x}^2\right\} - \gamma\left[\left(\frac{5h_{1,x}+2i\lambda_1}{2}\right)h_{1,xxx} + \frac{5h_{1,xx}^2}{4}\right] + 2\gamma\big(3\lambda_1^2 - h_{1,x}^2 - 3i\lambda_1h_{1,x}\big)h_{1,x}^2$$

$$m_x^{[0]} = \frac{1}{4}\big[(h_{1,x}+2i\lambda_1)^2 - v_1^2\big], \tag{9b}$$

and

$$r_1 = ih_{1,y} + 2\lambda_1 q_1 - (\alpha - 2\lambda_1\gamma)\left(s_1 + \frac{3}{2}h_{1,x}^2 - 6\lambda_1^2\right) + i\gamma\left\{h_{1,xxx} + 3h_{1,x}\left[s_1 + \frac{(h_{1,x}+2i\lambda_1)^2}{2}\right] + h_{1,x}^3\right\}, \tag{9c}$$

where we have defined $h_1 = \log(g_1)$ and r_1, v_1, q_1, s_1 are quantities that can be defined from ϕ_1 in the following form:

$$v_1 = \frac{\phi_{1,xx}}{\phi_{1,x}}, \quad r_1 = \frac{\phi_{1,t}}{\phi_{1,x}}, \quad q_1 = \frac{\phi_{1,y}}{\phi_{1,x}}, \quad s_1 = v_{1,x} - \frac{v_1^2}{2}. \tag{9d}$$

These definitions lead to the following relations:

$$v_{1,t} = (r_{1,x} + r_1 v_1)_x, \quad v_{1,y} = (q_{1,x} + q_1 v_1)_x. \tag{9e}$$

Equations (9a) and (9b) yield the expression of the fields in terms of the singular manifold. Furthermore, Eqs. (9c) and (9e) represent the singular manifold equations to be satisfied by ϕ_1 through the quantities (9d). It is important to remark that λ_1 is obtained by performing an integration in x and it is not a constant but a function of y and t satisfying the *nonisospectral condition*

$$\lambda_{1,t} - 2\lambda_1\lambda_{1,y} = 0. \tag{10}$$

C. Lax pair

In Refs. [15] and [16] it has been proved that the singular manifold is directly related to the eigenfunctions of the Lax operators. Actually, Eqs. (9a) can be easily linearized through the introduction of two functions ψ_1 and φ_1 defined as

$$v_1 = \frac{\psi_{1,x}}{\psi_1} + \frac{\varphi_{1,x}}{\varphi_1}, \quad h_{1,x} = \frac{\psi_{1,x}}{\psi_1} - \frac{\varphi_{1,x}}{\varphi_1}, \tag{11}$$

which allows (9a) to be rewritten as

$$\psi_{1,x} + u^{[0]}\varphi_1 + i\lambda_1\psi_1 = 0, \tag{12a}$$

$$\varphi_{1,x} + w^{[0]}\psi_1 - i\lambda_1\varphi_1 = 0. \tag{12b}$$

Therefore, we arrive at the spectral problem

$$\hat{L}\Psi = \lambda_1\Psi, \tag{13a}$$

where

$$\hat{L} = \begin{pmatrix} i\partial_x & iu^{[0]} \\ -iw^{[0]} & -i\partial_x \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \varphi_1 \end{pmatrix}. \tag{13b}$$

Moreover, the equation of motion of the eigenfunctions can be obtained by combining the singular manifold equations (9c) and (9e) with (9b). The result is

$$\psi_{1,t} = 2\lambda_1\psi_{1,y} + \lambda_{1,y}\psi_1 + i\left(m_y^{[0]}\psi_1 - u_y^{[0]}\varphi_1\right) + (\alpha - 2\lambda_1\gamma)F[\psi_1,\varphi_1,\lambda_1,u^{[0]},w^{[0]}], + i\gamma G[\psi_1,\varphi_1,u^{[0]},w^{[0]}], \tag{14a}$$

and

$$\varphi_{1,t} = 2\lambda_1\varphi_{1,y} + \lambda_{1,y}\varphi_1 - i\left(m_y^{[0]}\varphi_1 - w_y^{[0]}\psi_1\right) + (\alpha - 2\lambda_1\gamma)F[\varphi_1,\psi_1,\lambda_1,w^{[0]},u^{[0]}], - i\gamma G[\varphi_1,\psi_1,w^{[0]},u^{[0]}], \tag{14b}$$

where

$$F[\psi,\varphi,\lambda,u,w] = 3(uw + \lambda^2)\psi_x - \psi_{xxx} - 3u_x\varphi_x,$$
$$G[\psi,\varphi,u,w] = (3u^2w^2 + u_xw_x - uw_{xx} - wu_{xx})\psi + (6uwu_x - u_{xxx})\varphi. \tag{14c}$$

The combination of (9e) and (11) allows us to write the relation between the singular manifold ϕ_1 and the eigenfunctions ψ_1 and φ_1 as follows:

$$\phi_{1,x} = \psi_1\varphi_1, \quad \phi_{1,t} = 2\lambda_1\phi_{1,y} + i(\varphi_1\psi_{1,y} - \psi_1\varphi_{1,y}) + (\alpha - 2\lambda_1\gamma)J[\psi_1,\varphi_1,\lambda_1] + i\gamma K[\psi_1,\varphi_1,u^{[0]},w^{[0]}], \tag{15}$$

where we have defined

$$J[\psi,\varphi,\lambda] = 4\psi_x\varphi_x + 6\lambda^2\psi\varphi - \psi\varphi_{xx} - \varphi\psi_{xx},$$
$$K[\psi,\varphi,u,w] = \varphi\psi_{xx} - \psi\varphi_{xx} + 3(\psi_x\varphi_{xx} - \varphi_x\psi_{xx}) + 6uw(\psi\varphi_x - \varphi\psi_x).$$

III. DARBOUX TRANSFORMATIONS

According to results of the previous section, we can regard (8) as an iterative method such that an iterated solution $(u^{[1]}, w^{[1]}, m^{[1]})$ can be obtained from the seed solution $(u^{[0]}, w^{[0]}, m^{[0]})$ if we know a solution (ψ_1, φ_1) for the Lax pair of the seed solution with spectral parameter λ_1. This means that if we denote $(\psi_{1,2}, \varphi_{1,2})$, the eigenfunctions for $(u^{[1]}, w^{[1]}, m^{[1]})$ with eigenvalue λ_2, they should satisfy the Lax pair

$$(\psi_{1,2})_x + u^{[1]}\varphi_{1,2} + i\lambda_2\psi_{1,2} = 0,$$
$$(\varphi_{1,2})_x + w^{[1]}\psi_{1,2} - i\lambda_2\varphi_{1,2} = 0, \tag{16a}$$

and time derivatives

$$(\psi_{1,2})_t = 2\lambda_2(\psi_{1,2})_y + \lambda_{2,y}\psi_{1,2} + i\left(m_y^{[1]}\psi_{1,2} - u_y^{[1]}\varphi_{1,2}\right) + (\alpha - 2\lambda_2\gamma)F[\psi_{1,2},\varphi_{1,2},\lambda_2,u^{[1]},w^{[1]}] + i\gamma G[\psi_{1,2},\varphi_{1,2},u^{[1]},w^{[1]},],$$
$$(\varphi_{1,2})_t = 2\lambda_2(\varphi_{1,2})_y + \lambda_{2,y}\varphi_{1,2} - i\left(m_y^{[1]}\varphi_{1,2} - w_y^{[1]}\psi_{1,2}\right) + (\alpha - 2\lambda_2\gamma)F[\varphi_{1,2},\psi_{1,2},\lambda_2,w^{[1]},u^{[1]}] - i\gamma G[\varphi_{1,2},\psi_{1,2},w^{[1]},u^{[1]}]. \tag{16b}$$

The crucial point here is to consider the Lax pair as a set of nonlinear equations for the fields and eigenfunctions together [15,16]. It means that the expansion (8) should be accompanied by a similar expansion for the eigenfunctions, that in our case is

$$\psi_{1,2} = \psi_2 - \psi_1\frac{\Delta_{1,2}}{\phi_1}, \quad \varphi_{1,2} = \varphi_2 - \varphi_1\frac{\Delta_{1,2}}{\phi_1}, \tag{17}$$

where (ψ_i, φ_i) are eigenfunctions of the Lax pair for $(u^{[1]}, w^{[1]})$ with eigenvalue λ_j $(j = 1,2)$. Substitution of (17) into (14a), (16a), and (16b) trivially yields

$$\Delta_{1,2} = \frac{i}{2}\,\frac{\varphi_1\psi_2 - \varphi_2\psi_1}{\lambda_2 - \lambda_1}. \tag{18}$$

Equations (8) and (17) can be considered as transformations for the the Lax pair by using two sets (ψ_i,φ_i) with $i = 1,2$ of seed eigenfunctions. Therefore, they are binary Darboux transformations of the Lax pair and, according to (15), we can construct an iterated singular manifold $\phi_{1,2}$ satisfying

$$\begin{aligned}(\phi_{1,2})_x &= \psi_{1,2}\varphi_{1,2},\\ (\phi_{1,2})_t &= 2\lambda_2(\phi_{1,2})_y + i(\varphi_{1,2}(\psi_{1,2})_y - \psi_{1,2}(\varphi_{1,2})_y)\\ &\quad + (\alpha - 2\lambda_2\gamma)J[\psi_{1,2},\varphi_{1,2},\lambda_2]\\ &\quad + i\gamma K[\psi_{1,2},\varphi_{1,2},u^{[1]},w^{[1]}].\end{aligned} \tag{19}$$

The above equations can be viewed as a set of nonlinear equations involving fields, eigenfunctions, and singular manifolds that require a truncated expansion of $\phi_{1,2}$ of the form

$$\phi_{1,2} = \phi_2 - \frac{\Delta_{1,2}^2}{\phi_1}, \tag{20}$$

such that ϕ_i are singular manifolds for the seed solution $(u^{[0]},w^{[0]})$ with spectral parameters λ_i.

IV. τ FUNCTIONS

A. Iteration for the fields

In the previous section we have introduced a singular manifold $\phi_{1,2}$ which allows us to iterate (8) again in the following form:

$$\begin{aligned}u^{[2]} &= u^{[1]} + \frac{\psi_{1,2}^2}{\phi_{1,2}}, \quad w^{[2]} = w^{[1]} + \frac{\varphi_{1,2}^2}{\phi_{1,2}},\\ m^{[2]} &= m^{[1]} + \frac{(\phi_{1,2})_x}{\phi_{1,2}},\end{aligned} \tag{21}$$

that, combined with (8), (17), and (20), yields

$$\begin{aligned}u^{[2]} &= u^{[1]} + \frac{1}{\tau_{1,2}}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\begin{pmatrix}\Delta_{2,2} & -\Delta_{1,2}\\ -\Delta_{1,2} & \Delta_{1,1}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\\ w^{[2]} &= w^{[0]} + \frac{1}{\tau_{1,2}}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\begin{pmatrix}\Delta_{2,2} & -\Delta_{1,2}\\ -\Delta_{1,2} & \Delta_{1,1}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},\\ m^{[2]} &= m^{[0]} + \frac{(\tau_{1,2})_x}{\tau_{1,2}},\end{aligned} \tag{22}$$

where

$$\tau_{1,2} = \phi_{1,2}\phi_1 = \phi_2\phi_1 - \left(\Delta_{1,2}\right)^2 = \det(\Delta), \tag{23}$$

and Δ is a 2×2 matrix whose elements are given by

$$\Delta_{i,j} = \frac{i}{2}\,\frac{\varphi_i\psi_j - \varphi_j\psi_i}{\lambda_j - \lambda_i}, \quad i \neq j, \quad \Delta_{j,j} = \phi_j, \tag{24}$$

with $i,j = 1,2$ and

$$\begin{aligned}(\phi_j)_x = \psi_j\varphi_j, \quad \phi_{j,t} &= 2\lambda_j\phi_{j,y} + i(\varphi_j\psi_{j,y} - \psi_j\varphi_{j,y})\\ &\quad + (\alpha - 2\lambda_j\gamma)J[\psi_j,\varphi_j,\lambda_j]\\ &\quad + i\gamma K[\psi_j,\varphi_j,u^{[0]},w^{[0]}].\end{aligned} \tag{25}$$

B. Iteration for the eigenfunctions

A new iteration for the eigenfunctions can be written as

$$\begin{aligned}\psi_{1,2,3} &= \psi_{1,3} - \psi_{1,2}\frac{\Delta_{1,2,3}}{\phi_{1,2}},\\ \varphi_{1,2,3} &= \varphi_{1,3} - \varphi_{1,2}\frac{\Delta_{1,2,3}}{\phi_{1,2}},\\ \phi_{1,2,3} &= \phi_{1,3} - \frac{\Delta_{1,2,3}^2}{\phi_{1,2}},\end{aligned} \tag{26}$$

where $(\psi_{1,2,3},\varphi_{1,2,3})$ are eigenfunctions for $(u^{[2]},w^{[2]})$ with spectral parameter λ_3, and $(\psi_{1,i},\varphi_{1,i})$ are eigenfunctions for $(u^{[1]},w^{[1]})$ with spectral parameter λ_i. Straightforward calculations result in

$$\Delta_{1,2,3} = \Delta_{2,3} - \frac{\Delta_{1,2}\Delta_{1,3}}{\phi_1}, \tag{27}$$

and therefore

$$\begin{aligned}\psi_{1,2,3} &= \psi_3 + \frac{\psi_1}{\tau_{1,2}}[\Delta_{1,2}\Delta_{2,3} - \Delta_{1,3}\Delta_{2,2}]\\ &\quad + \frac{\psi_2}{\tau_{1,2}}[\Delta_{1,2}\Delta_{1,3} - \Delta_{2,3}\Delta_{1,1}],\end{aligned} \tag{28a}$$

$$\begin{aligned}\varphi_{1,2,3} &= \varphi_3 + \frac{\varphi_1}{\tau_{1,2}}[\Delta_{1,2}\Delta_{2,3} - \Delta_{1,3}\Delta_{2,2}]\\ &\quad + \frac{\varphi_2}{\tau_{1,2}}[\Delta_{1,2}\Delta_{1,3} - \Delta_{2,3}\Delta_{1,1}].\end{aligned} \tag{28b}$$

C. nth iteration

The above iteration brings about the definition

$$\tau_{1,2,3} = \phi_1\phi_{1,2}\phi_{1,2,3} = \det(\Delta), \tag{29}$$

where now the indices in (24) run $i,j = 1,2,3$. In general,

$$\tau_{1,2,\ldots,n} = \phi_1\phi_{1,2}\cdots\phi_{1,2,\cdots,n} = \det(\Delta), \tag{30}$$

such that the nth iteration for the fields (we write only the field m for simplicity) yields

$$m^{[n]} = m^{[0]} + \left[\frac{(\tau_{1,2,\ldots,n})_x}{\tau_{1,2,\ldots,n}}\right]_x. \tag{31}$$

We can summarize the iterative method of construction of solutions as follows:

(i) Solve the Lax pair in order to obtain the eigenfunctions (ϕ_i,φ_i) for a given seed solution $(u^{[0]},w^{[0]})$ and spectral parameter λ_i.

(ii) These eigenfunctions allow us to construct the $n\times n$ matrix Δ using (25).

(iii) The τ functions of order n can be obtained as the determinant of the matrix Δ.

(iv) The probability density $u^{[n]}w^{[n]} = -m_x^{[n]}$ can be easily obtained through (31).

The next section is devoted to the application of this method to obtain several interesting solutions which generalize previous results presented in Refs. [5,7,9,10].

V. LUMPS

With the above scenario in mind, it is worth discussing some specific solutions of the nonlinear system (4). We will focus on lumps, i.e., genuine solitons in $2+1$ dimensions that decay algebraically rather than exponentially with distance.

A. Seed solution and eigenfunctions

We consider a trivial seed solution of the form

$$u^{[0]} = i j_0, \quad w^{[0]} = -i j_0, \quad m^{[0]} = j_0^2 x - 3\gamma j_0^4 y, \quad (32)$$

where j_0 is an arbitrary constant. We can obtain rational solutions of the Lax pair as

$$\begin{aligned}
\psi_1 &= j_1 + j_0(j_2 - j_1) \\
&\quad \times \{x + b_1 j_0^2 y + 2[-3\alpha - i j_0^2(b_1 + 6\gamma)] j_0 t\}, \\
\varphi_1 &= j_2 + j_0(j_2 - j_1) \\
&\quad \times \{x + b_1 j_0^2 y + 2[-3\alpha + i j_0^2(b_1 + 6\gamma)] j_0 t\}, \\
\psi_2 &= j_2 + j_0(j_2 - j_1) \\
&\quad \times \{x + b_1 j_0^2 y + 2[-3\alpha - i j_0^2(b_1 + 6\gamma)] j_0 t\}, \\
\varphi_2 &= j_1 + j_0(j_2 - j_1) \\
&\quad \times \{x + b_1 j_0^2 y + 2[-3\alpha + i j_0^2(b_1 + 6\gamma)] j_0 t\}, \quad (33)
\end{aligned}$$

where $\lambda_1 = -\lambda_2 = i j_0$ and j_1, j_2, and b_1 are three arbitrary constants. This particular choice of λ_2 as the complex conjugate of λ_1 has been made in order to have ϕ_2 as the complex conjugate of ϕ_1 [16]. Our goal is to get a real τ function (23) without zeros and therefore solution (21) is nonsingular.

B. Matrix Δ

In this case, the solution of (24) can be written in terms of real functions A, B, C as

$$\phi_1 = A + iB, \quad \phi_2 = A - iB, \quad \Delta_{1,2} = -iC, \quad (34a)$$

where

$$\begin{aligned}
A &= 12\alpha h_1 j_0^7 t^2\{2\delta X + 3y(3\alpha^2 - j_0^2\delta^2)\} \\
&\quad - 2h_1 j_0^5 \delta t[X^2 + j_0^2 y^2(27\alpha^2 - j_0^2\delta^2)] \\
&\quad + 2h_2 j_0^4 t[\delta X + y(9\alpha^2 - j_0^2\delta^2)] \\
&\quad - h_1\alpha j_0 y[j_0^4 y^2(9\alpha^2 - 3j_0^2\delta^2) + 2] - 3h_2\alpha\delta j_0^4 y^2 \\
&\quad + 3\alpha j_0 j_1 j_2 y + \mathrm{Re}[Z], \quad (34b)
\end{aligned}$$

$$\begin{aligned}
B &= 4h_1 j_0^6 t^2\{X(9\alpha^2 - 3j_0^2\delta^2) - y\delta j_0^2(27\alpha^2 - j_0^2\delta^2)\} \\
&\quad - 6h_1\alpha j_0^4 t[X^2 + 3j_0^2 y^2(3\alpha^2 - j_0^2\delta^2)] \\
&\quad + 6h_2\alpha j_0^3 t(X - 2\delta j_0^2 y) \\
&\quad + \tfrac{1}{6}h_1 j_0^2[2X^3 + 2\delta j_0^4 y^3(27\alpha^2 - j_0^2\delta^2)] \\
&\quad + \tfrac{1}{6}h_1 j_0^2[y(7\delta - 4b_1)] \\
&\quad - \tfrac{1}{2}h_2 j_0[X^2 + j_0^2 y^2(9\alpha^2 - 3j_0^2\delta^2)] \\
&\quad + j_1 j_2(X - j_0^2\delta y) + \mathrm{Im}[Z], \quad (34c) \\
C &= \frac{h_1}{4j_0}[2j_0^2(6j_0^2\alpha t - X)^2 + 8j_0^8\delta^2 t^2 + 1] \\
&\quad + \frac{h_2}{2}(6j_0^2\alpha t - X) + \frac{j_1 j_2}{j_0}, \quad (34d)
\end{aligned}$$

and we have introduced the notation

$$\begin{aligned}
X &= x + b_1 j_0^2 y, \quad h_1 = (j_1 - j_2)^2, \\
h_2 &= j_1^2 - j_2^2, \quad \delta = 6\gamma + b_1, \quad (34e)
\end{aligned}$$

with $Z = Z(z)$ an arbitrary function of the variable $z = y + 2i j_0 t$ such that

$$Z_t - 2i j_0 Z_y = 0. \quad (34f)$$

C. τ function

According to (23) we have

$$\tau_{1,2} = A^2 + B^2 + C^2, \quad (35)$$

that it is a positive defined expression that never vanishes. Therefore, solutions (22) have no singularities. They depend on four arbitrary constants j_0, j_1, j_2, b_1 and an arbitrary function $Z(z)$, which means that there is a rich collection of possibilities for the solutions. We shall explore some of these solutions in the next section.

D. Lumps with $j_1 = j_2$

As an example, here we set $j_1 = j_2 = 1$ for simplicity and study different choices of the Z function.

1. Case $Z = 0$

The simplest case can be obtained by taking $Z = 0$. The eigenfunctions (33) are $\psi_1 = \psi_2 = \varphi_1 = \varphi_2 = 1$. Nevertheless, we have nontrivial expressions for ϕ_1, ϕ_2 and τ function. According to (34d), the expression for $\tau_{1,2}$ is

$$\tau_{1,2} = (3\alpha j_0 y)^2 + (X - j_0^2\delta y)^2 + \frac{1}{j_0^2}, \quad (36a)$$

and (22) yields

$$m^{[2]} = j_0^2 + \left[\frac{(\tau_{1,2})_X}{\tau_{1,2}}\right]_x, \quad (36b)$$

which is a static lump with a maximum at the point ($X = 0, y = 0$) and two minima at points ($X = \pm\sqrt{3}/2j_0, y = 0$). Figure 1(a) shows the static lump solution for this particular case ($j_1 = j_2 = 1$ and $Z = 0$).

2. Case $Z = a_1(y + 2i j_0 t)$ with a_1 real

For this particular choice of the Z function, (35) yields

$$\tau = (3j_0\alpha y + a_1 y)^2 + (X - j_0^2 y\delta + 2a_1 j_0 t)^2 + \frac{1}{4j_0^2}, \quad (37)$$

which is a lump that moves along the line $y = 0, X = -2a_1 j_0 t$.

3. Case $Z = a_1(y + 2i j_0 t)^2$ with a_1 real

In this case, Eq. (35) provides the following expression for the function τ:

$$\begin{aligned}
\tau &= (3j_0\alpha y + a_1 y^2 - 4a_1 j_0^2 t^2)^2 \\
&\quad + (X - j_0^2 y\delta + 4a_1 y j_0 t)^2 + \frac{1}{4j_0^2}. \quad (38)
\end{aligned}$$

When $t \to \pm\infty$, the solution behaves similarly to the motion of two lumps moving along the parabolas given by the

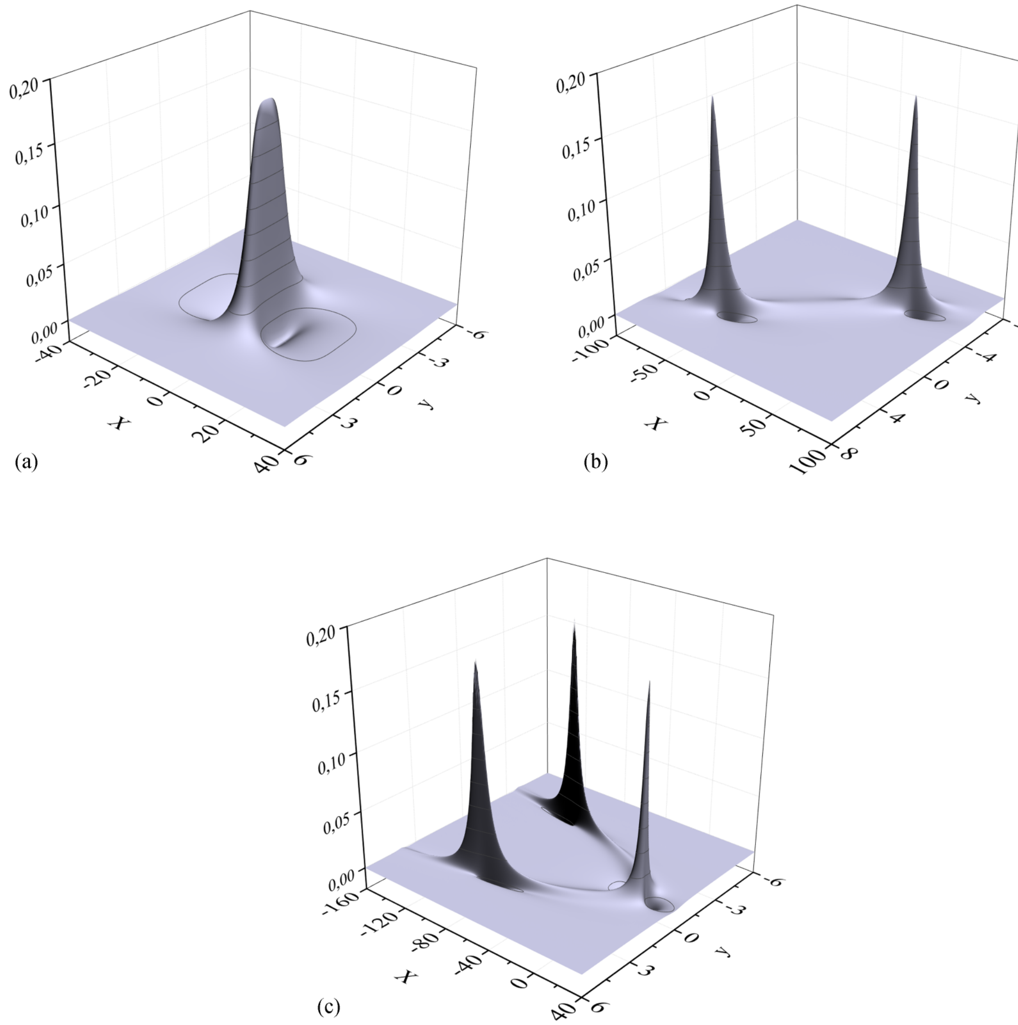

FIG. 1. Lump solitons when $j_1 = j_2 = 1$, $a_1 = 2$, $\alpha = 0.5$, and $\delta = 10$. (a) $Z = 0$ (static lump), (b) $Z = a_1(y + 2ij_0t)^2$, and (c) $Z = a_1(y + 2ij_0t)^3$ with $a_1 = 2$, and $j_0 = 0.15$. Lumps shown in (b) and (c) correspond to the asymptotic regime at $t > 0$.

parametric equations $y = \pm 2j_0t, X = \mp 8a_1j_0^2t^2$, as shown in Fig. 1(b).

4. Case $Z = a_1(y + 2ij_0t)^3$ with a_1 real

For this choice of the Z function (35) yields the real expression

$$\tau = (3j_0\alpha y + a_1y^3 - 12a_1yj_0^2t^2)^2 + (X - j_0^2y\delta + 6a_1y^2j_0t - 8a_1j_0^3t^3)^2 + \frac{1}{4j_0^2}. \quad (39)$$

Asymptotically, the solution behaves as two lumps moving along the cubic parabolas given by the parametric equations $y = \pm 2\sqrt{3}j_0t, X = -64a_1j_0^3t^3$ and a lump moving along the line $y = 0, X = 8a_1j_0^3t^3$, as seen in Fig. 1(c).

5. Case $Z = ia_1(y + 2ij_0t)^2$ with a_1 real

The τ function obtained from (35) reads

$$\tau = (3j_0\alpha y - 4a_1yj_0t)^2 + (X - j_0^2y\delta + a_1y^2 - 4a_1j_0^2t^2)^2 + \frac{1}{4j_0^2}. \quad (40)$$

The solution corresponds to a lump moving along the semiline $y = 0, X = 4a_1j_0^2t^2$. At $t = 0$ the solution goes to 0 at $y \to \infty$ as $8j_0^2/(36j_0^4\alpha^2y^2 + 1)$ along the parabola $X = j_0^2y\delta - a_1y^2$ [see Figs. 2(a) and 2(b)].

6. Case $Z = ia_1(y + 2ij_0t)^3$ with a_1 real

As a final example with $j_1 = j_2$, we consider the case $Z = ia_1(y + 2ij_0t)^3$. The τ function now reads

$$\tau = (3j_0\alpha y - 6a_1y^2j_0t + 8a_1j_0^3t^3)^2 + (X - j_0^2y\delta + a_1y^3 - 12a_1yj_0^2t^2)^2 + \frac{1}{4j_0^2}. \quad (41)$$

This solution behaves as two lumps moving along the parabola with parametric equations $y = (2\sqrt{3}/3)j_0t$, $X = \pm(64\sqrt{3}/9)a_1j_0^3t$. At $t = 0$ the solution goes to 0 at $y \to \infty$ as $8j_0^2/(1 + 36j_0^4\alpha^2y^2)$ along the cubic parabola $X = j_0^2\delta y - a_1y^3$ [see Figs. 2(c) and 2(d)].

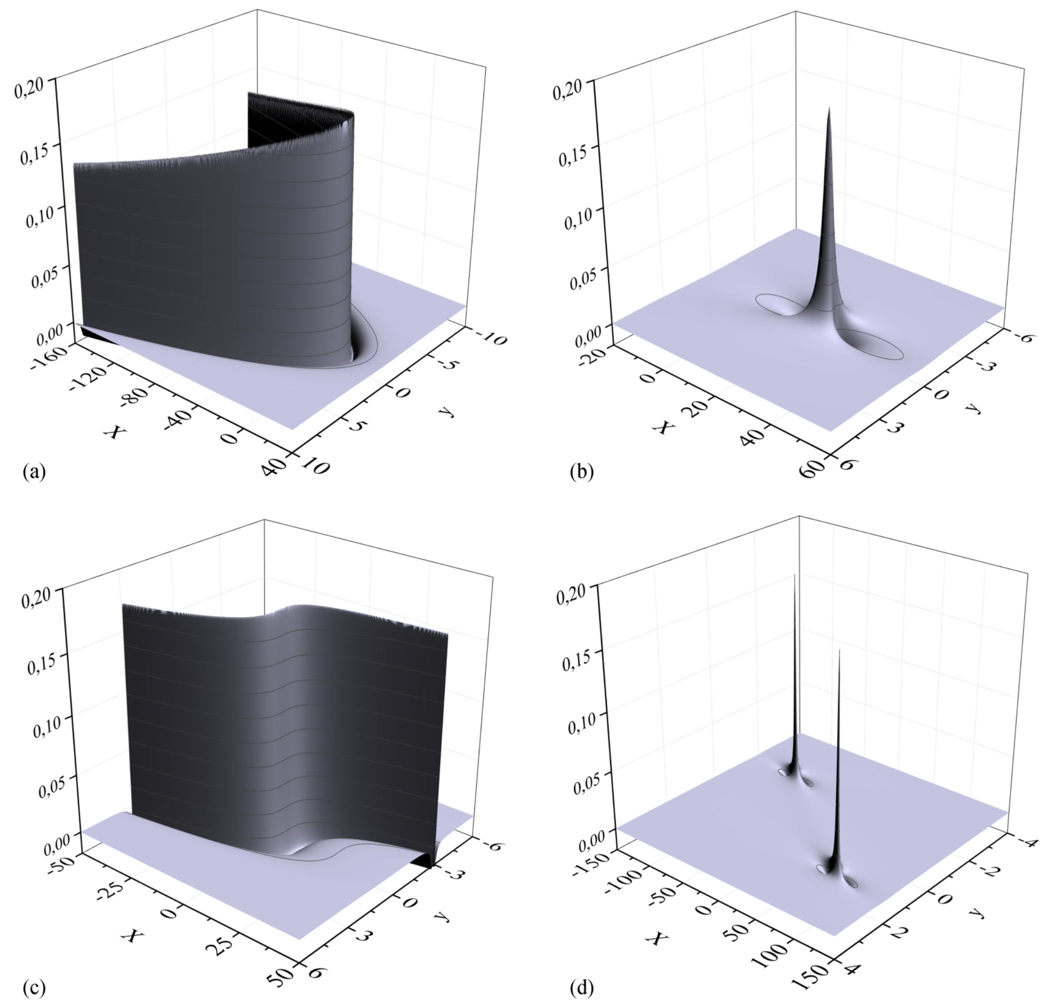

FIG. 2. Lump solitons at $t = 0$ (left panels) and in the asymptotic regime at $t > 0$ (right panels) when $j_1 = j_2 = 1$, $a_1 = 2$, $\alpha = 0.5$, and $\delta = 10$. Panels (a) and (b) correspond to $Z = ia_1(y + 2ij_0t)^2$ while (c) and (d) to $Z = ia_1(y + 2ij_0t)^3$.

E. Lumps with $j_1 \neq j_2$

Finally, we briefly discuss an example of the solutions found when $j_1 \neq j_2$. The simplest possibility corresponds to $Z = 0$. When $j_1 = -j_2 = 1$, the solution at $t \to \pm\infty$ behaves as a fixed lump in the origin and two lumps moving along the lines

$$X = \frac{2\sqrt{3}j_0^2(9\alpha^2 + j_0^2\delta^2)}{\sqrt{3}\alpha \pm j_0\delta},$$

$$y = \frac{2\sqrt{3}j_0(\pm\sqrt{3}\alpha - j_0\delta)}{\sqrt{3}\alpha \pm j_0\delta},$$

as can be seen in Fig. 3 in the asymptotic regime when $t > 0$.

VI. CONCLUSIONS

In conclusion, we have proposed and studied a generalization of the class of nonlinear evolution equations in $2 + 1$ dimensions originally introduced by Calogero [13] and later discussed by Zakharov [14]. The system of nonlinear equations (4) has the Painlevé property and, consequently, is, in this sense, integrable. The singular manifold method allows us to derive the Lax pair for this system. We have also presented an

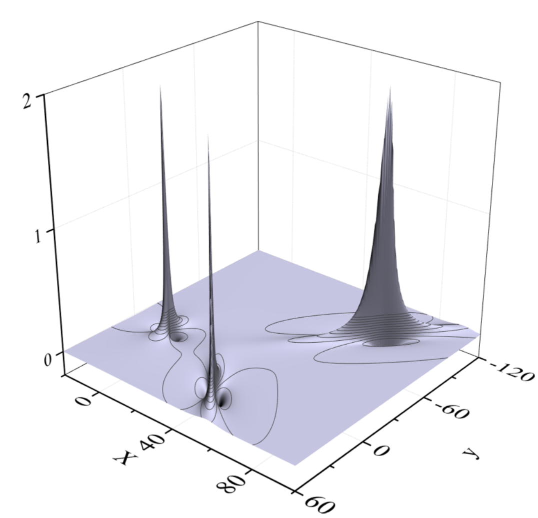

FIG. 3. Lump soliton in the asymptotic regime at $t > 0$ when $j_1 = -j_2 = 1$, $j_0 = 0.5$, $\alpha = 0.4$, $\delta = 1/2\sqrt{3}$, $b_1 = 1$, and $Z = 0$.

iterative procedure to build up lump solutions to the system of nonlinear equations (4). Most importantly, the proposed generalization involves third- and fourth-order dispersion terms that could pave the way to more elaborate models in the context of molecular energy transfer and nonlinear optics.

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