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Solvable Linear Potentials in the Dirac Equation.

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Abstract. – The Dirac equation for some linear potentials leading to Schrödinger-like oscillator equations for the upper and lower components of the Dirac spinor have been solved. Energy levels for the bound states appear in pairs, so that both particles and antiparticles may be bound with the same energy. For weak coupling, the spacing between levels is proportional to the coupling constant while in the strong limit those levels are depressed compared to the nonrelativistic ones.

Electromagnetic potentials are introduced in the Dirac equation according to the minimal substitution $p^\mu \to p^\mu - qA^\mu$. If the components of the Lorentz potential $A^\mu = (V, \boldsymbol{A})$ are independent of time and proportional to space coordinates (linearly rising potentials), one can describe the motion of Dirac particles under the action of constant electrostatic $\boldsymbol{E} = -\nabla V$ and magnetic $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ fields. Some special configurations of external potentials lead to solvable Dirac equations. In particular, solutions for electrons in homogeneous magnetic fields [1], homogeneous electrostatic fields [2], constant parallel [3] as well as crossed [4] electrostatic and magnetic fields have been found some years ago.

Since the advent of the quark model, linear potentials have renewed their interest because the confinement of quarks in mesons and baryons could be at least approximately explained, within the framework of phenomenological potential models. However, it is well established that electrostatic linear potentials lead to the occurrence of the Klein paradox and particles cannot be confined, no matter how large the electric field is. To obtain bound states, one must introduce a scalar potential by replacing m by $m + S$ in the Dirac equation, where S is also proportional to spatial coordinates. Hence, the rest mass of the particle increases indefinitely when the separation from the centre gets larger and larger, giving rise to confinement. Analytical solutions for the Dirac equation with linear scalar potentials are also found [5], even in addition to uniform electrostatic fields [6] or crossed, constant electrostatic and magnetic fields [7].

In a recent paper, Moshinsky and Szczepaniak [8] have introduced a new type of linear interaction in an attempt to describe a relativistic Dirac oscillator by means of an equation linear in both momenta and coordinates. The interaction is added to the wave equation by the substitution $\boldsymbol{p} \to \boldsymbol{p} - i\beta m\omega_s \boldsymbol{r}$, where ω_s is the coupling constant and β is a diagonal matrix defined below.

From the above considerations, the general form of the Dirac equation may be written in the standard notation as

$$[\boldsymbol{\alpha}\cdot(\boldsymbol{p}-i\beta\boldsymbol{u}+i\boldsymbol{v}-q\boldsymbol{A})+\beta(m+S)-(E-qV)]\psi=0 \tag{1}$$

$\boldsymbol{\alpha}$ and β being 4×4 matrices given by

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\qquad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},$$

where $\boldsymbol{\sigma}$ is the Pauli matrix vector and I stands for the 2×2 identity matrix. The interaction potential $\boldsymbol{\alpha}\cdot(-i\beta\boldsymbol{u}+i\boldsymbol{v}-q\boldsymbol{A})+\beta S+qV$ depends linearly on the coordinates. The aim of this letter is to solve some particular cases of eq. (1). We will be mainly focused on the conditions for the existence of bound states, so relativistic particles could be permanently confined. Hence the present work offers alternative forms of confining potentials which lead to exactly solvable Dirac equations. The mass m of the particle and the strength of the interaction terms appearing in (1) may be regarded as adjustable parameters to explain hadron spectra.

The first problem we consider is an interaction of the form $\boldsymbol{\alpha}\cdot(-i\beta m\omega_s\boldsymbol{r}-im\omega_v\boldsymbol{r})$. The Dirac equation (1) then reads

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m-E+im(\beta\omega_s+\omega_v)\,\boldsymbol{\alpha}\cdot\boldsymbol{r}]\psi=0\,. \tag{2}$$

This equation could be regarded as obtained from the free-particle equation by means of the replacement $E\rightarrow E-i\omega_v\,\boldsymbol{\alpha}\cdot\boldsymbol{r}$ and $m\rightarrow m+i\omega_s\,\boldsymbol{\alpha}\cdot\boldsymbol{r}$, in the same way as electrostatic and scalar interactions are added to the free-particle equation. The origin of the term proportional to $\boldsymbol{\alpha}\cdot\boldsymbol{r}$ may be easily understood as follows. Let us consider the Dirac equation with electrostatic and scalar harmonic-oscillator potentials

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta(m+(1/2)\,\gamma_s r^2)-(E-(1/2)\,\gamma_v r^2)]\,\psi=0.$$

By squaring it in standard fashion, one obtains that the Dirac spinor satisfies

$$[\boldsymbol{p}^2+(m+(1/2)\,\gamma_s r^2)^2-(E-(1/2)\,\gamma_v r^2)^2+i(\beta\gamma_s+\gamma_v)\,\boldsymbol{\alpha}\cdot\boldsymbol{r}]\,\psi=0,$$

which is a Klein-Gordon equation for harmonic potentials plus a nondiagonal term. This term, due to the interaction of the spin with electrostatic $\gamma_v\boldsymbol{r}$ and scalar $\gamma_s\boldsymbol{r}$ fields, is nothing but the interaction potential appearing in eq. (2). We consider, however, the interaction directly in the linear (Dirac) equation and not in the «squared» (Klein-Gordon) equation. Since the potential is invariant under spatial rotations, we seek for solutions of the form [9]

$$\psi=\frac{1}{r}\begin{pmatrix}if(r)\\ g(r)\,(\sigma\cdot\boldsymbol{r}/r)\end{pmatrix}\phi^l_{jm}, \tag{3}$$

where ϕ^l_{jm} denotes the normalized two-component eigenfunction of $\boldsymbol{J}^2$, J_z, $\boldsymbol{L}^2$ and S^2. The radial functions $f(r)$ and $g(r)$ satisfy two coupled first-order differential equations

$$(E-m)f(r)=\left\{-\frac{\mathrm{d}}{\mathrm{d}r}+\frac{K}{r}+m(\omega_s+\omega_v)\,r\right\}g(r)\,, \tag{4a}$$

$$(E+m)\,g(r)=\left\{\frac{\mathrm{d}}{\mathrm{d}r}+\frac{K}{r}+m(\omega_s-\omega_v)\,r\right\}f(r) \tag{4b}$$

with the notation

$$\kappa = \begin{cases} -(l+1) = -(j+1/2)\,, & j = l+1/2\,, \\ l = j+1/2\,, & j = l-1/2\,(l \neq 0)\,. \end{cases} \tag{5}$$

By decoupling (4*a*) and (4*b*) in standard fashion, we obtain for the upper radial function

$$\left\{-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + 2m\omega_{\mathrm{v}} r \frac{\mathrm{d}}{\mathrm{d}r} + \frac{l(l+1)}{r^2} + m^2(\omega_{\mathrm{s}}^2 - \omega_{\mathrm{v}}^2) r^2\right\} f(r) = [E^2 - m^2 - m\omega_{\mathrm{v}} - m\omega_{\mathrm{s}}(2\kappa - 1)] f(r) \tag{6}$$

and a similar equation is satisfied by the lower component $g(r)$. The regular solution at the origin is expressed in terms of confluent hypergeometric functions as

$$f(r) = A \exp[-m(\omega_{\mathrm{s}} - \omega_{\mathrm{v}}) r^2/2]\, r^{l+1} F\left(\frac{1}{2}(l + 3/2 - \mu),\ l + 3/2;\ m\omega_{\mathrm{s}} r^2\right), \tag{7}$$

where $\mu = [E^2 - m^2 - m\omega_{\mathrm{s}}(2\kappa - 1)]/2m\omega_{\mathrm{s}}$ and A is a constant evaluated by means of the normalization condition $\int_0^\infty \mathrm{d}r\,(|f(r)|^2 + |g(r)|^2) = 1$. Using the asymptotic behaviour of $f(r)$ at spatial infinity, it is an easy matter to check that only scattering states appear for $\omega_{\mathrm{s}} < \omega_{\mathrm{v}}$. On the contrary, $f(r)$ is square-integrable for $\omega_{\mathrm{s}} > \omega_{\mathrm{v}}$ thus representing bound states. The entire spectrum becomes discrete in this case. The corresponding energy levels are given by the quantization condition $(1/2)(l + 3/2 - \mu) = 0, -1, -2, \ldots$. Therefore one obtains

$$E_{Nj}^2 = m^2 + m\omega_{\mathrm{s}}(2N - 2j + 1) \tag{8a}$$

for $j = l + 1/2$ and

$$E_{Nj}^2 = m^2 + m\omega_{\mathrm{s}}(2N + 2j + 3) \tag{8b}$$

for $j = l - 1/2$ $(l \neq 0)$, N being a nonnegative integer. We might observe that bound levels are independent of ω_{v} and these levels coincide with those found by Moshinsky and Szczepaniak [8] in dealing with the Dirac oscillator. The radial eigenfunctions are found by using (4*b*) and (7). We get

$$f(r) = A r^{j+1/2} \exp[-m(\omega_{\mathrm{s}} - \omega_{\mathrm{v}}) r^2/2] F(-n_r, j+1;\ m\omega_{\mathrm{s}} r^2)\,, \tag{9a}$$

$$g(r) = -\frac{2m\omega_{\mathrm{s}} n_r A}{(j+1)(E+m)} r^{j+3/2} \exp[-m(\omega_{\mathrm{s}} - \omega_{\mathrm{v}}) r^2/2] F(-n_r + 1, j+2;\ m\omega_{\mathrm{s}} r^2) \tag{9b}$$

for $j = l + 1/2$ and

$$f(r) = A r^{j+3/2} \exp[-m(\omega_{\mathrm{s}} - \omega_{\mathrm{v}}) r^2/2] F(-n_r + 1, j+2;\ m\omega_{\mathrm{s}} r^2)\,, \tag{10a}$$

$$g(r) = \left(\frac{j+3/2}{E+m}\right) A r^{j+1/2} \exp[-m(\omega_{\mathrm{s}} - \omega_{\mathrm{v}}) r^2/2] F(-n_r + 1, j+1;\ m\omega_{\mathrm{s}} r^2) \tag{10b}$$

for $j = l - 1/2$, n_r being a positive integer.

The second situation we discuss is the Dirac oscillator with a constant crossed magnetic field, being represented by an interaction of the form $-\boldsymbol{\alpha} \cdot (q\boldsymbol{A} + i\beta \boldsymbol{u})$ in eq. (1). We consider the particle to be an electron ($q = -e$). Hence eq. (1) leads to

$$(E - m)\psi_u = \boldsymbol{\sigma} \cdot (\boldsymbol{p} + e\boldsymbol{A} + i\boldsymbol{u})\psi_l, \tag{11a}$$

$$(E + m)\psi_l = \boldsymbol{\sigma} \cdot (\boldsymbol{p} + e\boldsymbol{A} - i\boldsymbol{u})\psi_u, \tag{11b}$$

ψ_u and ψ_l being the upper and lower components of the wave functions, respectively. The interaction term $i\beta\boldsymbol{\alpha} \cdot \boldsymbol{u}$ describes a Dirac oscillator in the y-direction (say) and $\boldsymbol{A}$ is the vector potential of a constant crossed magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Therefore, we set the gauge

$$e\boldsymbol{A} = m\omega_c(y, 0, 0), \tag{12}$$

where $\omega_c = eB/m$ is the cyclotron frequency of the electron, and we also take

$$\boldsymbol{u} = m\omega_s(0, y, 0). \tag{13}$$

Since both $e\boldsymbol{A}$ and $\boldsymbol{u}$ are independent of x and z, the conjugate moments p_x and p_z are constants of motion. If the motion proceeds in the (X, Y)-plane, we can take $p_z = 0$. Using the *ansatz*

$$\psi_u = \exp[ip_x x] f(y)\phi_\lambda, \tag{14a}$$

$$\psi_l = \exp[ip_x x] g(y)\phi_{-\lambda}, \tag{14b}$$

where ϕ_λ is the two-component eigenvector of σ_z with eigenvalue $\lambda = \pm 1$, we obtain

$$\left\{-\frac{d^2}{dy^2} + [m(\omega_c + \lambda\omega_s)y \pm \lambda p_x]^2\right\}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} = [E^2 - m^2 \pm \lambda m(\omega_c + \lambda\omega_s)]\begin{pmatrix} f(y) \\ g(y) \end{pmatrix}. \tag{15}$$

Here the upper and lower signs refer to f and g, respectively. We note that eq. (15) reduces to the free-particle wave equation in the $z = 0$ plane for $\omega_s = \omega_c$ and $\lambda = -1$. For these particular values of the coupling constants there exist no bound states. In general cases, however, eq. (15) describes a one-dimensional nonrelativistic oscillator with coupling constant $\omega_c + \lambda\omega_s$. Energy levels are readily found to be

$$E_{n\lambda}^2 - m^2 = (2n + 1)|\omega_c + \lambda\omega_s| - \lambda(\omega_c + \lambda\omega_s), \tag{16}$$

n being an integer (zero or positive) whose lower value is chosen so that the right-hand side of (16) becomes positive-valued. Taking the limit $\omega_s \to 0$, the well-known Landau levels [10] for Dirac particles in homogeneous magnetic fields are recovered. The components of the corresponding eigenfunctions are Hermite polynomials times the usual exponential oscillator factor; we shall omit to write them down explicitly.

The last case we discuss in this letter is the one-dimensional Dirac oscillator in addition to electrostatic and scalar linear potentials. Dirac particles moving in one dimension may be described just by two-component spinors, rather than by the usual four-component spinors, since there is no spin-orbit coupling in one dimension [11], provided that the potential is spin-independent. The Dirac equation is simply written as

$$[\sigma_x(p - i\sigma_z u) + \sigma_z(m + S) + qV - E]\psi = 0 \tag{17}$$

in the standard representation. The potential depends linearly on the variable x

$$u = m\omega_s x, \quad S = k_s x, \quad qV = k_v x. \tag{18}$$

We now choose the *ansatz*

$$\psi = [\sigma_x(p - i\sigma_z u) + \sigma_z(m + S) - qV + E]\chi, \tag{19}$$

where the two-component spinor χ satisfies the following equation:

$$\{p^2 + (m^2\omega_s^2 + k_s^2 - k_v^2)x^2 + 2(mk_s + Ek_v)x + k_s\sigma_x + ik_v\sigma_y - m\omega_s\sigma_z\}\chi = (E^2 - m^2)\chi. \tag{20}$$

At large distance only the term proportional to x^2 becomes important, and hence eq. (20) reduces to a Schrödinger-like equation. Therefore, we easily conclude that bound states may only occur in the case $m\omega_s^2 + k_s^2 > k_v^2$, for which the coefficients of x^2 become positive. Moreover, the eigenvalues of the constant matrix $k_s\sigma_x + ik_v\sigma_y - m\omega_s\sigma_z$ are then real. These eigenvalues are $\lambda_\pm = \pm(m^2\omega_s^2 k_s^2 - k_v^2)^{1/2}$. Setting

$$\chi = f(x)\phi_\pm, \tag{21}$$

where $\phi_\pm$ is the eigenvector corresponding to the eigenvalue $\lambda_\pm$, we obtain

$$\{p_2 + (m_2\omega_s^2 + k_s^2 - k_v^2)x^2 + 2(mk_s + Ek_v)x\}f(x) = (E^2 - m^2 - \lambda_\pm)f(x), \tag{22}$$

which clearly reduces to a nonrelativistic oscillator equation by carrying out a suitable translation of the origin of coordinates. Bound levels are given by

$$E_n^2 - m^2 = 2(n+1)(m^2\omega_s^2 + k_s^2 - k_v^2)^{1/2} - (mk_s + Ek_v)^2(m^2\omega_s^2 + k_s^2 - k_v^2)^{-1}, \tag{23}$$

where n is a nonnegative integer. The explicit expression of the energy levels is easily found to be

$$E_n = \pm\left[\left(\frac{m^2\omega_s^2 + k_s^2 - k_v^2}{m^2\omega_s^2 + k_s^2}\right)\left(2(n+1)(m^2\omega_s^2 + k_s^2 - k_v^2)^{1/2} + \frac{m^4\omega_s^2}{m^2\omega_s^2 + k_s^2}\right)\right]^{1/2} - \frac{mk_s k_v}{m^2\omega_s^2 + k_s^2}. \tag{24}$$

Taking the limit $\omega_s \to 0$ in eq. (24) (*i.e.* neglecting the interaction term $i\sigma_z\sigma_x u$ in the Dirac Hamiltonian), we find the energy levels of a Dirac particle in a uniform electric field plus a linear scalar confining potential, in agreement with the results of Keng and Yuhong [6]. Eigenfunctions corresponding to the energy levels given by (24) are simply combinations of Hermite polynomials times an exponential oscillator factor, as obtained from (19), (21) and (22).

Some conclusions may be drawn from the above results. We have considered some linear interactions in the Dirac equation leading to exact solutions. When the coupling terms are carefully chosen, one obtains nonrelativistic harmonic-oscillator equations for the upper and lower components of the Dirac spinor, in spite of the linear dependence of the interaction potentials. This result provides a qualitative explanation why the nonrelativistic harmonic-oscillator quark model works so well for mesons and baryons [12]. When bound states appear, the energy levels are given by expressions of the form $E_k^2 = m^2 + kg$, k being a combination of quantum numbers and g depending on the coupling constants. Hence, energy levels always appear in pairs, which means that particles as well as antiparticles may be bound with the same energy. The lowest-lying energy levels can be approximately given by $E_k - m \simeq kg$ in the weak-coupling limit, so the spacing between levels is proportional to the coupling constant, as occurs in the nonrelativistic harmonic oscillator. On the contrary, if g is much larger than the rest mass, E_k rises as the square root of the coupling constant, and levels are then depressed from the nonrelativistic prediction. This results agrees with more elaborated treatments based on the Bethe-Salpeter equation [13]. Regarding the spherically symmetric potentials appearing in (2), we have demonstrated that $E^2 \simeq j$ for large values of the angular momentum, as seen from (8) and (9). Hence the Regge trajectories are asymptotically like those of the nonrelativistic harmonic oscillator. The same result is also valid for the Dirac equation with vector and scalar potentials $qV(r) = k_v r$ and $S(r) = k_s r$, as pointed out by Fishbane *et al.* [14]. Nevertheless, the Dirac equation for such potentials cannot be exactly solved, and one must invoke BKW or variational methods. Finally, let us comment that the effects of Coulomb-like potentials [15], which may appear between quarks caused by exchange of massless gluons, can be qualitatively described replacing the actual r^{-1} potential by a more simplified short-ranged potential. This is thought to be a good approximation since at distance from the centre only the linear potential term becomes important. Successful short-ranged potentials are the one-dimensional and the spherically symmetric delta-function potentials, for which the Dirac equation is also exactly solvable [16, 17]. In fact, this realization has been recently considered in harmonic quark models with mixed vector plus scalar harmonic-oscillator potentials [18]. We shall report on the exact solutions of the Dirac equation with solvable linear interactions plus delta-function potentials in more detail elsewhere.

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