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A one-dimensional relativistic screened Coulomb potential

F. Domínguez-Adame, A. Rodríguez

Departamento de Física de Materiales, Facultad de Físicas, Universidad Complutense, 28040 Madrid, Spain

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Abstract

We propose a screened Coulomb potential leading to an exactly solvable one-dimensional Dirac equation. Unlike the one-dimensional Coulomb potential, the screened potential can support truly bound states because the Klein paradox is absent, provided that the potential does not dive into the negative-energy continuum. The δ-function limit of the potential is considered in detail. In the conclusions we discuss possible applications of our results in different physical contexts.

The non-relativistic Schrödinger equation for screened Coulomb potentials provides a useful description of heavy quark physics [1]. In regards to lighter quarkonia, relativistic effects can be included in a non-perturbative way by solving the Klein–Gordon equation [2]. Moreover, the solution of many other physical problems where relativistic effects could play an important role (e.g. solid state physics [3]) are largely simplified using screened potentials because singularities of form factors at the origin are smeared out. In this way, Banerjee and Chakravorty [4] have investigated the scattering solutions of the Dirac equation for a short-ranged potential, simulating the effects of a screened Coulomb-type interaction. On the other hand, Domínguez-Adame [5] has studied relativistic effects on the bound states of the Hulthén potential, another kind of short-ranged potential often used in molecular and solid state physics.

In many cases of interest, quantum mechanical equations for the three-dimensional screened potential cannot be found analytically (see Refs. [6,7] and references therein). This is more dramatic when dealing with the Dirac equation for some potentials (exponential, Hulthén, Yukawa, Morse), where the potential barrier due to the spin of the particle does not allow us to find analytical solutions even for s states. On the other hand, one-dimensional potentials can provide exact solutions to shed some light on the problem. Because of this, the one-dimensional Dirac equation has been attracting much attention recently [8–11].

One of the most useful screened potentials is the Yukawa potential. It behaves like the Coulomb potential close to the origin but decreases exponentially at large distances. The form factor for the Yukawa potential is $F(p) = (p^2 + a^{-2})^{-1}$, where a is the screening distance of the potential. In the theory of meson exchange, this value is inversely proportional to the mass of the exchange particle. The one-dimensional potential presenting the same form factor is the exponential potential $V(x) = -(g/2a)\exp(-|x|/a)$, where g denotes the coupling constant. Close to the origin, it becomes of the form $V(x) \sim -g/2a + (g/2a^2)|x|$. Besides the constant background potential $-g/2a$, this limit is the solution of the Poisson equation for a point charge in a one-dimensional space. On the other hand, far from the origin $|x| \gg \hat{a}$ it vanishes exponentially. Hence this potential

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could be regarded as a screened Coulomb potential in one dimension. In the present paper we obtain the steady state solutions of the Dirac equation ($\hbar = c = 1$)

$$\left(\mathrm{i}\alpha\frac{\mathrm{d}}{\mathrm{d}x} + \beta[E - V(x)] - m\right)\boldsymbol{\phi}(x) = 0, \tag{1}$$

for an electrostatic-like (the time component of a four Lorentz vector) screened Coulomb potential. Here ϕ denotes the two-component particle wave function

$$\boldsymbol{\phi}(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}. \tag{2}$$

For convenience, we set the following representation for the Dirac matrices,

$$\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

Therefore, Eq. (1) reads as follows,

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + E + \frac{g}{2a}\,\mathrm{e}^{-x/a}\right)\phi_2(x) = m\phi_1(x), \tag{4a}$$

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + E + \frac{g}{2a}\,\mathrm{e}^{-x/a}\right)\phi_1(x) = m\phi_2(x), \tag{4b}$$

for $x > 0$ and

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + E + \frac{g}{2a}\,\mathrm{e}^{x/a}\right)\phi_2(x) = m\phi_1(x), \tag{5a}$$

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + E + \frac{g}{2a}\,\mathrm{e}^{x/a}\right)\phi_1(x) = m\phi_2(x), \tag{5b}$$

for $x < 0$.

We will be concerned with bound state solutions. Hence, $\boldsymbol{\phi}(x)$ should decrease at infinity in a suitable way,

$$\lim_{|x|\to\infty} \phi_1(x) = \lim_{|x|\to\infty} \phi_2(x) = 0. \tag{6}$$

We start with the solution for positive values of x. Inserting (4b) in (4a) one obtains the following Schrödinger-like equation for the upper component

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - q^2 + \frac{g}{2a}\left(2E + \frac{\mathrm{i}}{a}\right)\mathrm{e}^{-x/a} + \left(\frac{g}{2a}\right)^2 \mathrm{e}^{-2x/a}\right]\phi_1(x) = 0, \tag{7}$$

where $x > 0$ and $q = (m^2 - E^2)^{1/2}$ are real for bound states. Solutions vanishing at infinity can be written in terms of regular confluent hypergeometric functions $M(\alpha, \beta; z)$. After using Eq. (4b), the particle wave function for $x > 0$ is found to be

$$\boldsymbol{\phi}(x) = A_+ \exp(-qx - \tfrac{1}{2}\mathrm{i}g\,\mathrm{e}^{-x/a}) \begin{pmatrix} M(1 + qa - \mathrm{i}Ea, 1 + 2qa; -\mathrm{i}g\,\mathrm{e}^{-x/a}) \\ [(E - \mathrm{i}q)/m]M(qa - \mathrm{i}Ea, 1 + 2qa; -\mathrm{i}g\,\mathrm{e}^{-x/a}) \end{pmatrix}, \tag{8a}$$

A_+ being an integration constant. To find the solution for negative x we can use the symmetry of the Dirac equation. To be specific, replacing x, ϕ_1 and ϕ_2 in Eqs. (4) by $-x$, ϕ_2 and ϕ_1, respectively, we obtain Eqs. (5), so the solution for $x < 0$ is readily written as

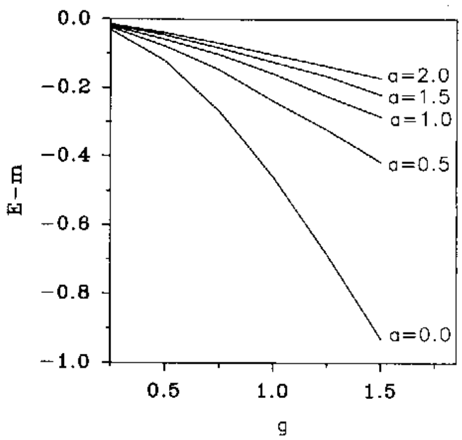

Fig. 1. Ground state energy level for a particle of mass unity as a function of the coupling constant g for several values of the range of the screened Coulomb potential.

$$\phi(x) = A_- \exp(qx - \tfrac{1}{2}\mathrm{i}g\,\mathrm{e}^{x/a}) \begin{pmatrix} [(E-\mathrm{i}q)/m]M(qa - \mathrm{i}Ea, 1+2qa; -\mathrm{i}g\,\mathrm{e}^{x/a}) \\ M(1+qa-\mathrm{i}Ea, 1+2qa; -\mathrm{i}g\,\mathrm{e}^{x/a}) \end{pmatrix}, \tag{8b}$$

with A_- an integration constant which, of course, is related to A_+ requiring the continuity of the wave function at $x = 0$.

Notice that $\phi(x)$ decreases exponentially as $|x| \to \infty$, being square-integrable and thus representing a truly bound state. Therefore, the one-dimensional screened Coulomb potential we have introduced can actually bind particles. This is not a trivial result in view of the fact that the Dirac equation for the one-dimensional Coulombic coupling (linear electrostatic-like potential) equation only presents scattering states because electrostatic linearly rising potentials polarize the vacuum and create electron–positron pairs, no matter how small the coupling constant is [12–15]. This phenomenon is exactly analogous to the well-known Klein paradox.

The corresponding bound state levels are found by imposing the continuity of the wave function at the origin, as we mentioned before. Using Eqs. (8) and the Kummer transformation $M(\alpha,\beta;z) = \mathrm{e}^z\, M(\beta-\alpha,\beta;-z)$ (see Ref. [16]) we get

$$\left(\frac{E-\mathrm{i}q}{m}\right)^2 = \frac{M(qa+\mathrm{i}Ea, 1+2qa; \mathrm{i}g)M(1+qa-\mathrm{i}Ea, 1+2qa; -\mathrm{i}g)}{M(qa-\mathrm{i}Ea, 1+2qa; -\mathrm{i}g)M(1+qa+\mathrm{i}Ea, 1+2qa; \mathrm{i}g)}. \tag{9}$$

Introducing the notation $\lambda_k = \lambda_k(E,g) = \arg[M(k+qa+\mathrm{i}Ea, 1+2qa; \mathrm{i}g)]$ where $k = 0, 1$, we finally obtain

$$q = E\tan(\lambda_1 - \lambda_0) \tag{10}$$

for the energy levels. This transcendental equation has to be solved by the usual search methods. It is worth mentioning the existence of an underlying symmetry that explains how the one-dimensional screened Coulomb potential can bind particles or antiparticles. This is clearly seen taking into account that $\lambda_k(-E,-g) = -\lambda_k(E,g)$, so that using (10) the following symmetry is obtained,

$$E(g) = -E(-g). \tag{11}$$

Hence, reversing the sign of the coupling constant, the opposite energy spectrum is reached. Therefore, we can restrict ourselves to study only the case $g > 0$. Fig. 1 shows the ground state level for different values of the coupling constant g and the range of the potential a for a particle with $m = 1$. As it should be expected, for a given value of a the bound state becomes deeper on increasing the coupling constant g. On the other hand, for a fixed value of g, the bound state becomes shallower on increasing a because the range of the potential als appears in the pre-exponential factor, and actually the effective minimum value of the potential is $-g/2a$.

Let us now discuss the limit of a very short-ranged potential ($a \to 0$). In such a case, $V(x)$ approaches th δ-function limit of the form $-g\delta(x)$. One way to solve the Dirac equation for a sharply peaked potential i to find the solution for a square well (or barrier), and then allow the width to go down to zero while keepin the *area* of the well constant [17]. As we have already mentioned, the screened Coulomb potential with vanishing screening distance is an alternative way to study this limit. From the definition of λ_k it is clear tha $\lambda_0 = 0$ and $\lambda_1 = g$ as $a \to 0$. Therefore, we find a single bound state for particles whose energy is given by th relationship $q = E \tan g$. The same expression is obtained by solving the Dirac equation for any sharply peake potential [18,19], as it should be. The corresponding energy level is labelled $a = 0$ in Fig. 1. It is seen that th δ-function potential can bind particles stronger than finite-ranged potentials.

Finally, we discuss briefly the physical contexts where our results may be of interest. On the one hand, it i usually difficult to study the quantum dynamics of relativistic fermions under short-ranged interactions exactl and results must restore on numerical or perturbative solutions, as we mentioned above. One-dimensiona potentials bring the opportunity to obtain exact results which are easily interpreted, while keeping the essentia physics. On the other hand, our present results may also be useful in semiconductor physics in view of th analogy existing between the Dirac equation and the two-band model Hamiltonians of semiconductors [20]. I particular, in the study of nonparabolicity effects in δ-doped semiconductors, which can be described in practic by a potential analogous to the above-introduced screened Coulomb potential [21], one is faced with a simila problem as the one we have solved in the present paper. Hence we believe that our work is beyond the forma solution of the Dirac equation for a particular screened Coulomb potential and it should be of great interest i a variety of physical situations.

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