Casimir Forces in Classical and Quantum Systems

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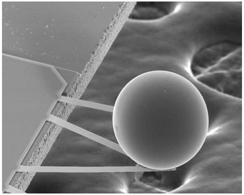
Esquema

Objectives

1. Present a new method to calculate Casimir forces valid in equilibrium and out of equilibrium systems.
2. Verification of known results of the literature with this method
3. New results in non-equilibrium systems.

Historical Introduction

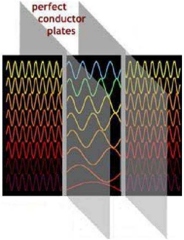
- Quantum Vacuum induces an atractive force between uncharged plates because vacuum fluctuations of em field: Casimir, 1948. Proc. K. Ned. Akad. Wet. 51, 793
- It has recently been measured with great precision: Mohideen and Roy, 1998. PRL, 81, 4549



Historical Introduction

Non compensation of vacuum density of energy out and between the plates, there are *more* modes in than out the plates:

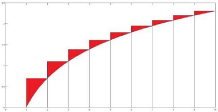
$$\text{Inside:} \quad k_n = \frac{2\pi}{L} n \quad \forall n \in \mathbb{Z}$$
$$\text{Outside:} \quad k_n = \frac{2\pi}{L} n \quad \forall n \in \mathbb{R}$$



Both vacuum energies diverge, but their difference is *finite*:

$$\langle E \rangle_{in} = \frac{\hbar c}{2} \sum_{n \in \mathbb{Z}} k_n \to \infty$$

$$\langle E \rangle_{out} = \frac{\hbar c}{2} \int_{-\infty}^{\infty} k(n) dn \to \infty$$



$$\boxed{\langle E \rangle_{in} - \langle E \rangle_{out} = -\frac{\hbar c \pi^2}{720 L^3}}$$

Fluctuations induced Casimir Forces

Casimir forces appears on systems where:

1. Fluctuations of any kind of origin appears.

$$p(\mathbf{r}) \rightarrow \langle p(\mathbf{r}) \rangle$$

2. Broken translation symmetry.

$$E \rightarrow \langle E \rangle \Rightarrow \langle \delta E \rangle = 0$$

$$E \rightarrow \langle E \rangle \Rightarrow \langle \delta E(\mathbf{r}) \rangle = -\sum_i \langle F_i(\mathbf{r}) \rangle \, \delta \mathbf{r}_i$$

3. There are long range correlations.

Fuerzas de Casimir inducidas por fluctuaciones

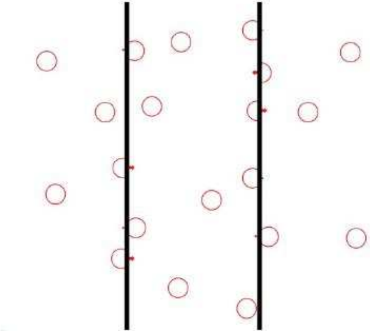


Figura: Short range

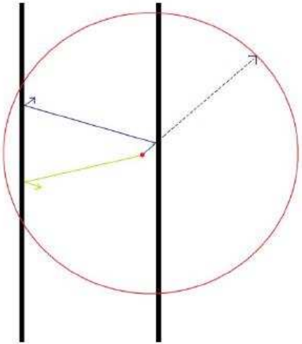


Figura: Long range

Pressure over the bodies

Due to fluctuations, we take the system from equilibrium to a steady state.

$$p(\mathbf{r}) \rightarrow \langle p(\mathbf{r}) \rangle$$

$$\langle p(\mathbf{r}) \rangle = \langle p(\phi_0) \rangle + \frac{\delta p(\phi_0)}{\delta \phi(\mathbf{r})} \langle \phi(\mathbf{r}) \rangle + \frac{1}{2} \frac{\delta^2 p(\phi_0)}{\delta \phi(\mathbf{r}) \delta \phi(\mathbf{r}')} \langle \phi(\mathbf{r}) \phi(\mathbf{r}') \rangle$$

$$\langle p(\mathbf{r}) \rangle = \langle p(\partial_i \phi_0) \rangle + \frac{\delta p(\partial_i \phi_0)}{\delta \partial_i \phi(\mathbf{r})} \langle \partial_i \phi(\mathbf{r}) \rangle + \frac{1}{2} \frac{\delta^2 p(\partial_i \phi_0)}{\delta \partial_i \phi(\mathbf{r}) \delta \partial_i \phi(\mathbf{r}')} \langle \partial_i \phi(\mathbf{r}) \partial_i \phi(\mathbf{r}') \rangle$$

1. Pressure mean does not contribute, because: $\langle p(\phi_0(\mathbf{r})) \rangle = p(\phi_0(\mathbf{r}))$
2. Steady state over the noise: $\langle \phi(\mathbf{r}) \rangle = \langle \partial_i \phi(\mathbf{r}) \rangle = 0$
3. Just if we have long range correlations: $\langle \phi^2(\mathbf{r}) \rangle \neq 0 \qquad \left\langle (\partial_i \phi(\mathbf{r}))^2 \right\rangle \neq 0$

Pressure over the bodies

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$$\langle p(\mathbf{r})\rangle = \langle p(\partial_i\phi_0)\rangle + \frac{\delta p(\partial_i\phi_0)}{\delta\partial_i\phi(\mathbf{r})}\langle\partial_i\phi(\mathbf{r})\rangle + \frac{1}{2}\frac{\delta^2 p(\partial_i\phi_0)}{\delta\partial_i\phi(\mathbf{r})\delta\partial_i\phi(\mathbf{r}')}\langle\partial_i\phi(\mathbf{r})\partial_i\phi(\mathbf{r}')\rangle$$

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Pressure over the bodies

Due to fluctuations, we take the system from equilibrium to a steady state.

$$p(\mathbf{r}) \longrightarrow \langle p(\mathbf{r}) \rangle$$

$$\langle p(\mathbf{r}) \rangle = \langle p(\phi_0) \rangle + \frac{\delta p(\phi_0)}{\delta \phi(\mathbf{r})} \langle \phi(\mathbf{r}) \rangle + \frac{1}{2} \frac{\delta^2 p(\phi_0)}{\delta \phi(\mathbf{r}) \delta \phi(\mathbf{r}')} \langle \phi(\mathbf{r}) \phi(\mathbf{r}') \rangle$$

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Pressure over the bodies

Casimir force:

$$\mathbf{F} = \oint ds \langle p(\mathbf{s}) \rangle = \frac{1}{2} \frac{\delta^2 p(\phi_0)}{\delta\phi(\mathbf{r})\delta\phi(\mathbf{r})} \oint ds \left\langle \phi^2(\mathbf{s}) \right\rangle$$

The integral diverges over the whole integration interval, so it is a bad defined integral

$$\left\langle \phi^2(\mathbf{s}) \right\rangle \to \infty$$

But these divergences are compensated when we integrate over the whole surface. Divergences are compensated!

$$\oint ds \left\langle \phi^2(\mathbf{s}) \right\rangle < \infty$$

Conclussion: Casimir forces depends of the field correlations over the noise.

Calculating averages over the noise

Field without fluctuations:

$$\partial_t \phi - L\phi = 0$$

Field with fluctuations:

$$\partial_t \phi - L\phi = \xi(\mathbf{r}, t)$$

General solution:

$$\phi(\mathbf{r}, t) = \frac{1}{V} \sum_{n \in \mathbb{Z}} \phi_n(t) f_n(\mathbf{r})$$

Where:

$$\begin{pmatrix} Lf_n(\mathbf{r}) = L_n f_n(\mathbf{r}) \\ f_n[\text{surface}] = 0 \end{pmatrix} \Rightarrow \{f_n(\mathbf{r})\}_{n \in \mathbb{Z}}$$

Calculating averages over the noise

Stationary average of the field over the noise:

$$\left\langle \phi^2(\mathbf{r}) \right\rangle = \lim_{t\to\infty} \frac{1}{V^2} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \langle \phi_n(t)\phi_m^*(t) \rangle \, f_n(\mathbf{r}) f_m^*(\mathbf{r})$$

$$\left\langle (\partial_i\phi(\mathbf{r}))^2 \right\rangle = \lim_{t\to\infty} \frac{1}{V^2} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \langle \phi_n(t)\phi_m^*(t) \rangle \, \partial_i f_n(\mathbf{r}) \partial_i f_m^*(\mathbf{r})$$

Field mode correlations because the noise:

$$\lim_{t\to\infty} \langle \phi_n(t)\phi_m^*(t) \rangle = V S_{nm}$$

Calculating averages over white noise

$$\left\langle \xi\left(\mathbf{r},t\right)\xi^*\left(\mathbf{r}',t'\right)\right\rangle = \Gamma\delta\left(\mathbf{r}-\mathbf{r}'\right)\delta\left(t-t'\right) \Rightarrow S_{nm} = \frac{\Gamma}{2L_n}\delta_{nm}$$

Averages to calculate:

$$\boxed{\left\langle \phi^2(\mathbf{r})\right\rangle = \frac{\Gamma}{2V}\sum_{n\in\mathbb{Z}}\frac{f_n^*(\mathbf{r})f_m(\mathbf{r})}{L_n} = \frac{\Gamma}{2}\lim_{\mathbf{r}'\to\mathbf{r}} G(\mathbf{r},\mathbf{r}')}$$

$$\boxed{\left\langle (\partial_i\phi(\mathbf{r}))^2\right\rangle = \frac{\Gamma}{2V}\sum_{n\in\mathbb{Z}}\frac{\partial_i f_n^*(\mathbf{r})\partial_i f_m(\mathbf{r})}{L_n} = \frac{\Gamma}{2}\lim_{\mathbf{r}'\to\mathbf{r}} \partial_i\partial_i' G(\mathbf{r},\mathbf{r}')}$$

- Principal result.
- Valid for any kind of geometry.
- In principle, valid for any noise and a general kind of fields.
- Fluctuation-Dissipation theorem let us reobtain known results of the literature.

Calculating averages over colored noise

$$\left\langle \xi\left(\mathbf{r},t\right)\xi^{*}\left(\mathbf{r}',t'\right)\right\rangle = \Gamma h\left(\mathbf{r}-\mathbf{r}'\right)F\left(t-t'\right)$$

Then:

$$\boxed{S_{nm} = \frac{\Gamma}{L_n + L_n}\int d\mathbf{r}\int d\mathbf{r}' f_n(\mathbf{r}) h\left(\mathbf{r}-\mathbf{r}'\right) f_m^{*}(\mathbf{r}')\int_0^{\infty} F(t)e^{-L_n t}dt}$$

Classical Problems

We are going to study the parallel plates geometries in these cases:

1. Thermal Casimir effect in Reaction-Difussion environment.
2. Thermal Casimir effect in dissipative nematic liquid cristals.
3. Electromagnetic Quantum Casimir effect.
4. Maximal space correlated noise.
5. Time correlated noise.

Reaction-Difussion environment

Reaction-Difussion dynamical equation and thermal noise:

$$\partial_t \phi - D\Delta\phi + \lambda\phi = \xi(\mathbf{r}, t) \qquad \langle \xi(\mathbf{r}, t)\, \xi(\mathbf{r}_0, t_0) \rangle = 2k_B T \delta(\mathbf{r} - \mathbf{r}_0)\, \delta(t - t_0)$$

No flux cross the plates $\Rightarrow \partial_x \phi(0) = \partial_x \phi(L) = 0$
Pressure depends on the square of the field:

$$\frac{\delta^2 p(\phi_0)}{\delta\phi(\mathbf{r})\delta\phi^*(\mathbf{r})} = \frac{\delta^2}{\delta\phi(\mathbf{r})\delta\phi^*(\mathbf{r})} T^{xx} = 1$$

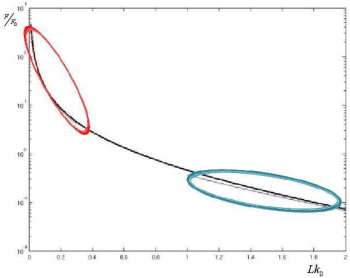
Averaged pressure over the plates:

$$\langle p \rangle = \frac{1}{2} \left\langle \phi^2 \right\rangle_{in} - \frac{1}{2} \left\langle \phi^2 \right\rangle_{out}$$

Reaction-Difussion environment

$$\langle p \rangle = \frac{k_B T}{4\pi D L} \log\left(1 - e^{-Lk_0}\right)$$

Correlation length: $\zeta = k_0^{-1} = \sqrt{\frac{D}{\lambda}}$



Short range correlations:

$$L \gg \zeta \Rightarrow \langle p \rangle = -\frac{k_B T}{4\pi D L} e^{-Lk_0}$$

Long range correlations:

$$L \ll \zeta \Rightarrow \langle p \rangle = \frac{k_B T}{4\pi D L} \log(Lk_0)$$

Dissipative nematic liquid cristals

Dissipative nematic liquid cristals dynamics and thermal noise:

$$\partial_t \phi - D\Delta\phi + \lambda\phi = \xi(\mathbf{r}, t) \qquad \langle \xi(\mathbf{r}, t)\, \xi(\mathbf{r}_0, t_0) \rangle = 2k_B T \delta(\mathbf{r} - \mathbf{r}_0)\, \delta(t - t_0)$$

Strong Nematic boundary conditions $\Rightarrow \phi(0) = \phi(L) = 0$
Pressure depends on the square of the field:

$$\frac{\delta^2 p(\partial_x \phi_0)}{\delta\partial_x\phi(\mathbf{r})\delta\partial_x\phi^*(\mathbf{r})} = \frac{\delta^2}{\delta\partial_x\phi(\mathbf{r})\delta\partial_x\phi^*(\mathbf{r})} T^{xx} = 1$$

Averaged pressure over the plates:

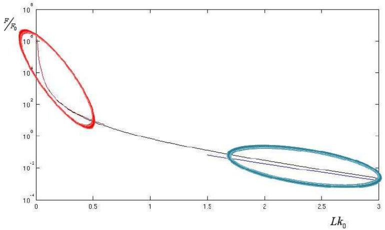
$$\langle p \rangle = \frac{1}{2}\left\langle (\partial_x\phi)^2 \right\rangle_{in} - \frac{1}{2}\left\langle (\partial_x\phi)^2 \right\rangle_{out}$$

Dissipative nematic liquid cristals

$$\langle p \rangle = -\frac{k_B T}{8\pi D L^3}\left[Li_3\left(e^{-2Lk_0}\right) + 2Lk_0 Li_2\left(e^{-2Lk_0}\right) + 2(Lk_0)^2 Li_1\left(e^{-2Lk_0}\right)\right]$$

Where $Li_s(z) \sum_{n=1}^{\infty} \frac{z^n}{n^s}$

Correlation length: $\zeta = k_0^{-1} = \sqrt{\frac{D}{\lambda}}$



Short range correlations:

$$L \gg \zeta \Rightarrow \langle p \rangle = -k_B T \frac{k_0^3}{\pi D} \frac{e^{-2Lk_0}}{4Lk_0}$$

Long range correlations:

$$L \ll \zeta \Rightarrow \langle p \rangle = -\frac{k_B T}{8\pi D L^3}\zeta(3)$$

Quantum Electromagnetic field

Quantum Electromagnetic field dynamic and F-D theorem:

$$\frac{1}{c^2}\partial_t^2\phi - \Delta\phi = \xi(\mathbf{r}, t) \qquad \langle \phi^*(\mathbf{r}, t)\,\phi(\mathbf{r}_0, t_0)\rangle = \frac{\hbar}{2} G(\mathbf{r} - \mathbf{r}_0)$$

Perfect metal boundary conditions $\Rightarrow \phi(0) = \phi(L) = 0$
Pressure depends on the square of the gradient of the field:

$$\frac{\delta^2 p(\partial_x\phi_0)}{\delta\partial_x\phi(\mathbf{r})\delta\partial_x\phi^*(\mathbf{r})} = \frac{\delta^2}{\delta\partial_x\phi(\mathbf{r})\delta\partial_x\phi^*(\mathbf{r})} T^{xx} = 1$$

Averaged pressure over the plates:

$$\langle p\rangle = \frac{1}{2}\left\langle(\partial_x\phi)^2\right\rangle_{in} - \frac{1}{2}\left\langle(\partial_x\phi)^2\right\rangle_{out}$$

$$\left\langle (\partial_x \phi)^2 \right\rangle_{in} = -\frac{\hbar c}{AL} \int d\mathbf{k} \sum_{n\in\mathbb{Z}} \frac{\left(\frac{\pi n}{L}\right)^2}{\sqrt{\left(\frac{\pi n}{L}\right)^2 + \mathbf{k}^2}}$$

$$\left\langle (\partial_x \phi)^2 \right\rangle_{out} = -\frac{\hbar c}{AL} \int d\mathbf{k} \int_{-\infty}^{\infty} \frac{\left(\frac{\pi n}{L}\right)^2}{\sqrt{\left(\frac{\pi n}{L}\right)^2 + \mathbf{k}^2}} dn$$

Then we reobtain:

$$\boxed{\langle p \rangle = -\frac{\hbar c \pi^2}{240 L^4}}$$

Maximal space correlated noise

Dynamics and maximal space correlated noise:

$$\partial_t \phi - D\Delta\phi + \lambda\phi = \xi(\mathbf{r}, t) \qquad \langle \xi(\mathbf{r}, t)\, \xi(\mathbf{r}_0, t_0) \rangle = \Gamma \delta(t - t_0)$$

Dirichlet boundary conditions $\Rightarrow \phi(0) = \phi(L) = 0$
Averaged pressure over the plates:

$$S_{nm} = \frac{2\Gamma}{L_n + L_m} \frac{V_{\parallel}^2}{k_{nx} k_{mx}} \delta^d_{\vec{n},\vec{0}} \delta^d_{\vec{m},\vec{0}} [1 - (-1)^n][1 - (-1)^m]$$

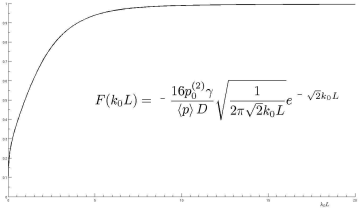
$$\left\langle (\partial_x \phi)^2 \right\rangle = \frac{4\Gamma}{DL^2} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \frac{[1 - (-1)^n]\,[1 - (-1)^m]}{\left(\frac{\pi}{L} n\right)^2 + \left(\frac{\pi}{L} m\right)^2 + 2k_0^2}$$

Maximal space correlated noise

The Casimir force is:

$$\langle p \rangle = \partial^2_{\partial_x \phi} p \frac{4\Gamma}{D\pi^3} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}}^{(n,m)\neq(0,0)} \left[\begin{array}{l} K_0\left(2\sqrt{2}k_0 L\sqrt{n^2+m^2}\right) \\ -K_0\left(2\sqrt{2}k_0 L\sqrt{\frac{n^2}{4}+m^2}\right) \\ +\frac{1}{4}K_0\left(\sqrt{2}k_0 L\sqrt{n^2+m^2}\right) \end{array} \right].$$

$$\lim_{k_0 L \gg 1} \langle p \rangle = -\partial^2_{\partial_x \phi} p \frac{16\Gamma}{D\pi} \sqrt{\frac{\pi}{2\sqrt{2}k_0 L}} e^{-\sqrt{2}k_0 L}$$



Time correlated noise

Dynamics and temporal correlated noise:

$$\partial_t \phi - D\Delta\phi + \lambda\phi = \xi(\mathbf{r}, t) \qquad \langle \xi(\mathbf{r}, t)\, \xi(\mathbf{r}_0, t_0)\rangle = \Gamma\delta(\mathbf{r} - \mathbf{r}_0)\, a^\alpha e^{-|a|(t-t_0)}$$

Dirichlet boundary conditions $\Rightarrow \phi(0) = \phi(L) = 0$
Averaged pressure over the plates:

$$S_{nm} = \frac{a^\alpha}{L_n + a} \frac{\Gamma}{2L_n} \delta_{nm}$$

$$\left\langle (\partial_n \phi(0))^2 \right\rangle = \frac{a^\alpha \Gamma}{V} \sum_{n\in\mathbb{Z}} \frac{1}{L_n + a} \frac{k_{nx}^2}{L_n} = \frac{a^{\alpha-1}\Gamma}{V} \sum_{n\in\mathbb{Z}} \left[\frac{k_{nx}^2}{L_n} - \frac{k_{nx}^2}{L_n + a} \right]$$

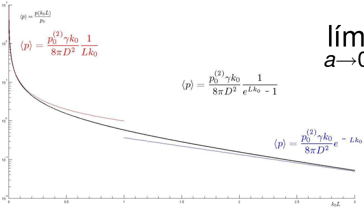
Time correlated noise

The Casimir force is:

$$\langle p \rangle = p_0'' \frac{a^{\alpha-1}\Gamma}{8\pi D L_x^3} \left[\begin{array}{l} Li_3(e^{-2L_x k_0}) + 2k_0 L_x Li_2(e^{-2L_x k_0}) + 2k_0^2 L_x^2 Li_1(e^{-2L_x k_0}) \\ -Li_3(e^{-2L_x k_1}) - 2k_1 L_x Li_2(e^{-2L_x k_1}) - 2k_1^2 L_x^2 Li_1(e^{-2L_x k_1}) \end{array} \right]$$

$$k_0^2 = \frac{\lambda}{D} \qquad\qquad k_1^2 = \frac{\lambda}{D} + \frac{a}{D}$$

Maximal correlated temporal noise:



$$\lim_{a\to 0} \lim_{\alpha=0} \langle \xi(\mathbf{r}, t)\, \xi(\mathbf{r}_0, t_0) \rangle = \Gamma \delta(\mathbf{r} - \mathbf{r}_0)$$

$$\lim_{a\to 0} \langle p \rangle = -\frac{p_0''}{2} \frac{\Gamma k_0}{4\pi D^2} \frac{1}{e^{Lk_0} - 1}$$

Conclussions

- New method to calculate Casimir forces from dynamical equations of the fields.
- We recober literature known results in the equilibrium cases.
- New results for non-equilibrium systems.
- Three conditions to obtain Casimir forces.
- Casimir effect is not just a quantum or thermal effect, it is more general, linked to any stochastic system.
- Importance of long range correlations.

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THANKS FOR YOUR ATTENTION