Bogoliubov Excitations of Disordered Bose-Einstein Condensates

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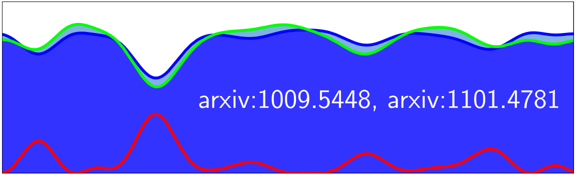


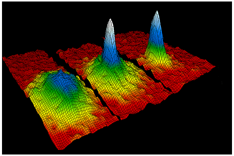
GISC workshop, February 18, 2011

Bose-Einstein Condensation

of atomic gases

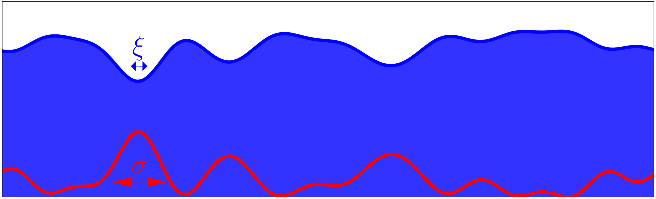
- Bose occupation number $n_i = \dfrac{1}{e^{(\varepsilon_i - \mu)/k_\mathrm{B}T} - 1} \to \infty,$ for $\mu \nearrow \varepsilon_0$
- Critical point:
  thermal deBroglie wave-length λ_T
  exceeds particle distance: $n\lambda_T^d \gtrsim 1$
- Macroscopic occupation
  of single-particle ground state $\Psi(\boldsymbol{r})$

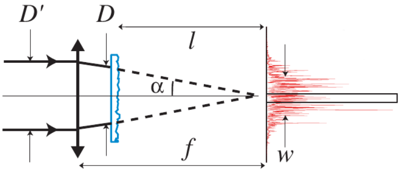
[http://www.colorado.edu/physics/2000/bec/]



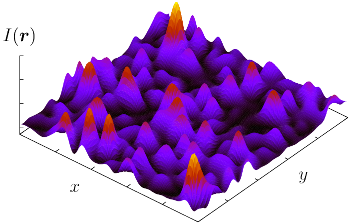
Optical disorder potentials

Laser speckle





[Clément et al., NJP **8**, 165 (2006)]

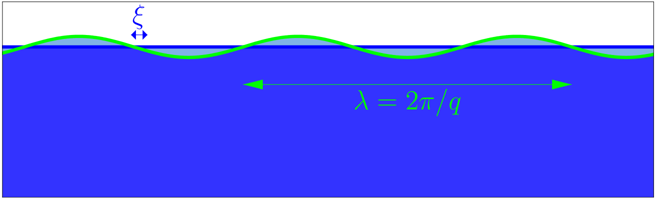


[R. Kuhn, PhD thesis, U Bayreuth (2007)]

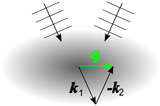
- $\overline{V_{\boldsymbol{k}} V_{-\boldsymbol{k}'}} = \delta_{\boldsymbol{k}\boldsymbol{k}'} \tilde{R}(|\boldsymbol{k}|\sigma)$

Bogoliubov Excitations

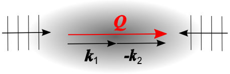
Accessible via Bragg spectroscopy

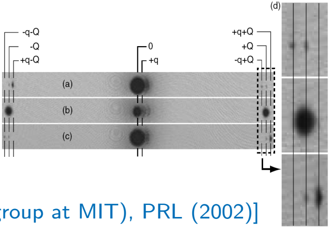


- Excitation of $\hat{\gamma}_{\boldsymbol{q}}$



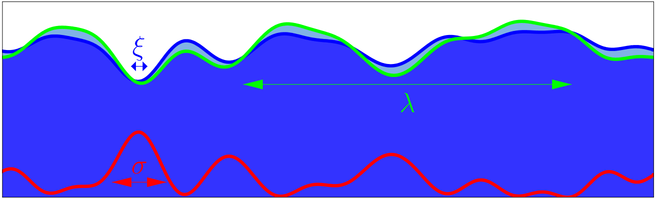
- Detection





[Vogels *et al.* (Ketterle group at MIT), PRL (2002)]

Disordered Bogoliubov problem



Many-body Hamiltonian and Bogoliubov approach

$$E[\hat{\Psi}, \hat{\Psi}^\dagger] = \int \mathrm{d}^d r\, \hat{\Psi}^\dagger(\boldsymbol{r}) \left[\frac{-\hbar^2}{2m} \nabla^2 + V(\boldsymbol{r}) + \frac{g}{2} \hat{\Psi}^\dagger(\boldsymbol{r}) \hat{\Psi}(\boldsymbol{r}) - \mu \right] \hat{\Psi}(\boldsymbol{r})$$

$$\left[\hat{\Psi}(\boldsymbol{r}), \hat{\Psi}(\boldsymbol{r}')^\dagger \right] = \delta(\boldsymbol{r} - \boldsymbol{r}'), \quad \ldots$$

Bose-Einstein condensation

- Ideal BEC: condensation into single-particle ground state
  $\Rightarrow$ Bose broken symmetry: $\langle \hat{\Psi}(\boldsymbol{r}) \rangle = \Psi(\boldsymbol{r})$

Bogoliubov prescription

$$\hat{\Psi}(\boldsymbol{r}) = \Psi(\boldsymbol{r}) + \delta\hat{\psi}(\boldsymbol{r})$$

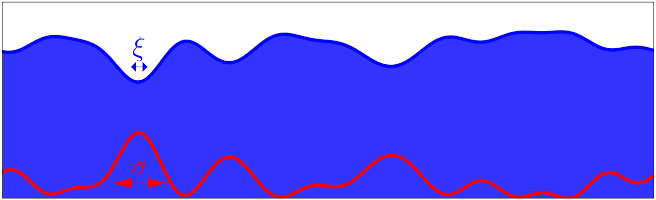
- condensate density $|\Psi(\boldsymbol{r})|^2$
- condensate depletion $\langle \delta\hat{\psi}^\dagger \delta\hat{\psi} \rangle$

Gross-Pitaevskii mean-field $\hat{\Psi}(\boldsymbol{r}) = \Psi(\boldsymbol{r}) + \cancel{\delta\hat{\psi}(\boldsymbol{r})}$

$E[\Psi, \Psi^*]$ classical energy functional

$$\frac{\delta E}{\delta \Psi^*} = -\frac{\hbar^2}{2m}\nabla^2\Psi(\boldsymbol{r}) + \left[V(\boldsymbol{r}) + g\,|\Psi(\boldsymbol{r})|^2 - \mu\right]\Psi(\boldsymbol{r}) = 0$$

healing length $\xi = \frac{\hbar}{\sqrt{2mgn_0}}$



$\Psi(x)^2$, $V(x)$, $\sigma \gg \xi$

Gross-Pitaevskii mean-field $\hat{\Psi}(\boldsymbol{r}) = \Psi(\boldsymbol{r}) + \delta\hat{\psi}(\boldsymbol{r})$

$E[\Psi, \Psi^*]$ classical energy functional

$$\frac{\delta E}{\delta \Psi^*} = -\frac{\hbar^2}{2m}\nabla^2\Psi(\boldsymbol{r}) + \left[V(\boldsymbol{r}) + g\,|\Psi(\boldsymbol{r})|^2 - \mu\right]\Psi(\boldsymbol{r}) = 0$$

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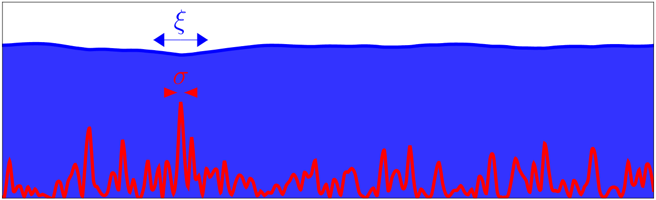


$\Psi(x)^2$, $V(x)$, $\sigma \ll \xi$

Back to the fluctuations: $\hat{\Psi}(\boldsymbol{r}) = \Psi(\boldsymbol{r}) + \boxed{\delta\hat{\psi}(\boldsymbol{r})}$

Expansion of the Hamiltonian (in powers of $\delta\hat{\psi}$)

$$E[\Psi + \delta\hat{\psi}, \ldots] = E_0 + 0 + \underbrace{\iint \mathrm{d}^d r' \mathrm{d}^d r \left\{ \frac{\delta^2 E}{\delta\Psi \delta\Psi'} \delta\hat{\psi}(\boldsymbol{r}) \delta\hat{\psi}(\boldsymbol{r}') + \ldots \right\}}_{\hat{H}} + \ldots$$

Density and phase

$$\delta\hat{n}(\boldsymbol{r}) = \Psi(\boldsymbol{r}) \Big\{ \delta\hat{\psi}(\boldsymbol{r}) + \delta\hat{\psi}^\dagger(\boldsymbol{r}) \Big\}, \quad i\Psi(\boldsymbol{r})\delta\hat{\varphi}(\boldsymbol{r}) = \frac{1}{2} \Big\{ \delta\hat{\psi}(\boldsymbol{r}) - \delta\hat{\psi}^\dagger(\boldsymbol{r}) \Big\}$$

$$\big[\delta\hat{n}(\boldsymbol{r}), \delta\hat{\varphi}(\boldsymbol{r}')\big] = i\delta(\boldsymbol{r} - \boldsymbol{r}'), \quad \ldots$$

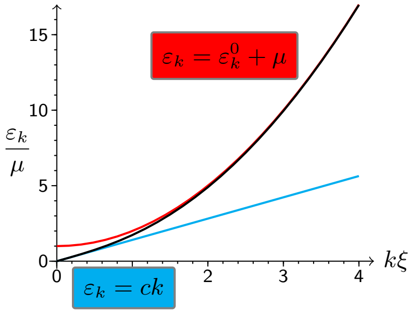
Excitation spectrum of the homogeneous BEC

$$\hat{H}^{(0)} = \sum_{\boldsymbol{k}} \left[\varepsilon_k^0 n_0 \delta\hat{\varphi}_{\boldsymbol{k}} \delta\hat{\varphi}_{-\boldsymbol{k}} + (\varepsilon_k^0 + 2\mu) \frac{\delta\hat{n}_{\boldsymbol{k}} \delta\hat{n}_{-\boldsymbol{k}}}{4n_0} \right], \quad \varepsilon_k^0 = \frac{\hbar^2 k^2}{2m}$$

$$\hat{H}^{(0)} = \sum_{\boldsymbol{k}} \varepsilon_k \hat{\gamma}_{\boldsymbol{k}}^\dagger \hat{\gamma}_{\boldsymbol{k}}$$

$$\varepsilon_k = \sqrt{\varepsilon_k^0 (2\mu + \varepsilon_k^0)}$$

$$\hat{\gamma}_{\boldsymbol{k}} = \sqrt{\frac{\varepsilon_k}{\varepsilon_k^0}} \frac{\delta\hat{n}_{\boldsymbol{k}}}{2\Psi_0} + i \sqrt{\frac{\varepsilon_k^0}{\varepsilon_k}} \Psi_0 \delta\hat{\varphi}_{\boldsymbol{k}}$$



Condensate depletion $\left\langle \delta\hat{\psi}^\dagger \delta\hat{\psi} \right\rangle_0$ of the homogeneous BEC

- Bosonic excitations:

$$\left[\hat{\gamma}_{\boldsymbol{k}}, \hat{\gamma}^\dagger_{\boldsymbol{k}'}\right] = \delta_{\boldsymbol{k}\boldsymbol{k}'} \qquad\qquad \hat{\gamma}_{\boldsymbol{k}} \left|0\right\rangle = 0$$

- $\delta\hat{\psi}_{\boldsymbol{k}}$ contains $\hat{\gamma}^\dagger_{-\boldsymbol{k}}$ as well:

$$\delta\hat{\psi}_{\boldsymbol{k}} = \frac{1}{2}\left\{\sqrt{\frac{\varepsilon_k}{\varepsilon^0_k}} + \sqrt{\frac{\varepsilon^0_k}{\varepsilon_k}}\right\}\hat{\gamma}_{\boldsymbol{k}} + \frac{1}{2}\left\{\sqrt{\frac{\varepsilon_k}{\varepsilon^0_k}} - \sqrt{\frac{\varepsilon^0_k}{\varepsilon_k}}\right\}\hat{\gamma}^\dagger_{-\boldsymbol{k}}$$

- Thus

$$\frac{\delta n^{(0)}}{n} = \frac{\left\langle \delta\hat{\psi}^\dagger \delta\hat{\psi} \right\rangle_0}{|\Psi^2|} = \frac{1}{4}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\left\{\sqrt{\frac{\varepsilon_k}{\varepsilon^0_k}} - \sqrt{\frac{\varepsilon^0_k}{\varepsilon_k}}\right\}^2 \overset{\text{3D}}{=} \frac{8}{3\sqrt{\pi}}\sqrt{n a_{\mathrm{s}}^3}$$

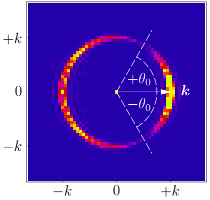
- Condensate depletion scales with the dilute-gas parameter $\sqrt{n a_{\mathrm{s}}^3}$

Inhomogeneous Bogoliubov Hamiltonian

$V(\boldsymbol{r}) \to \Psi(\boldsymbol{r}) \to$ *Scattering*

$$\hat{H}[\hat{\gamma},\hat{\gamma}^\dagger] = \sum_{\boldsymbol{k}} \varepsilon_k \hat{\gamma}^\dagger_{\boldsymbol{k}}\hat{\gamma}_{\boldsymbol{k}} + \underbrace{\frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{k}'} \left(\hat{\gamma}^\dagger_{\boldsymbol{k}'},\hat{\gamma}_{-\boldsymbol{k}'}\right) \begin{pmatrix} W_{\boldsymbol{k}'\boldsymbol{k}} & Y_{\boldsymbol{k}'\boldsymbol{k}} \\ Y_{\boldsymbol{k}'\boldsymbol{k}} & W_{\boldsymbol{k}'\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} \hat{\gamma}_{\boldsymbol{k}} \\ \hat{\gamma}^\dagger_{-\boldsymbol{k}} \end{pmatrix}}_{=: \frac{1}{2}\hat{\Gamma}^\dagger \mathcal{V}\,\hat{\Gamma}}$$

- Normal scattering $W_{\boldsymbol{k}'\boldsymbol{k}} \sim V_{\boldsymbol{k}'-\boldsymbol{k}}$
  [CG & C.A. Müller, EPL **83**, 10006 (2008)]

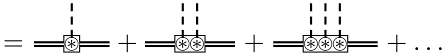


- Anomalous coupling $Y_{\boldsymbol{k}'\boldsymbol{k}}$

  $\to$ Nambu notation: $\hat{\Gamma}_{\boldsymbol{k}} = \begin{pmatrix} \hat{\gamma}_{\boldsymbol{k}} \\ \hat{\gamma}^\dagger_{-\boldsymbol{k}} \end{pmatrix}$

- Expansion (in powers of V/μ)

$$\mathcal{V} = \mathcal{V}^{(1)} + \mathcal{V}^{(2)} + \mathcal{V}^{(3)} + \dots$$



Disorder-averaged excitation spectrum

Retarded Green function

$$\mathcal{G}_{\boldsymbol{k}\boldsymbol{k}'}(t) := \frac{1}{i\hbar}\left\langle \begin{pmatrix} [\hat{\gamma}_{\boldsymbol{k}}(t), \hat{\gamma}^{\dagger}_{\boldsymbol{k}'}(0)] & [\hat{\gamma}_{-\boldsymbol{k}}(t), \hat{\gamma}_{\boldsymbol{k}'}(0)] \\ [\hat{\gamma}^{\dagger}_{-\boldsymbol{k}}(t), \hat{\gamma}^{\dagger}_{\boldsymbol{k}'}(0)] & [\hat{\gamma}^{\dagger}_{\boldsymbol{k}}(t), \hat{\gamma}_{\boldsymbol{k}'}(0)] \end{pmatrix} \right\rangle \Theta(t)$$

Equation of motion

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} i\hbar \frac{\mathrm{d}}{\mathrm{d}t} - [\mathcal{H}_0 + \mathcal{V}] \right\} \mathcal{G} = \mathbb{1}, \qquad (\mathcal{H}_0)_{\boldsymbol{k}'\boldsymbol{k}} = \varepsilon_k \delta_{\boldsymbol{k}'\boldsymbol{k}}$$

Homogeneous Green function

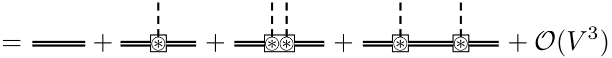
$$\mathcal{G}_0(\omega) = \begin{pmatrix} G_0(\omega) & 0 \\ 0 & G_0^*(-\omega) \end{pmatrix}, \qquad G_0(\omega) = \frac{1}{\hbar\omega - \varepsilon_k + i0}$$

Disorder expansion

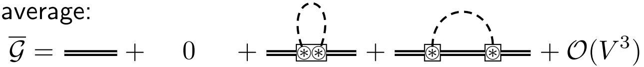
$$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0 \mathcal{V} \mathcal{G}_0 + \mathcal{G}_0 \mathcal{V} \mathcal{G}_0 \mathcal{V} \mathcal{G}_0 + \ldots$$

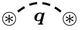
Disorder-averaged effective medium

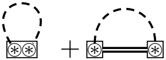
$$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0 \mathcal{V} \mathcal{G}_0 + \mathcal{G}_0 \mathcal{V} \mathcal{G}_0 \mathcal{V} \mathcal{G}_0 + \dots$$



Disorder average:



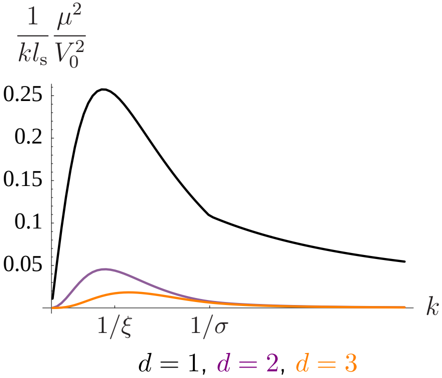
Sum over disorder correlator  $= \overline{|V_{\boldsymbol{q}}|^2}$

Self-energy $\Sigma^{(2)} =$ 

$$\hbar\omega = \varepsilon_k + \Sigma^{(2)}_{11}(k,\omega)$$

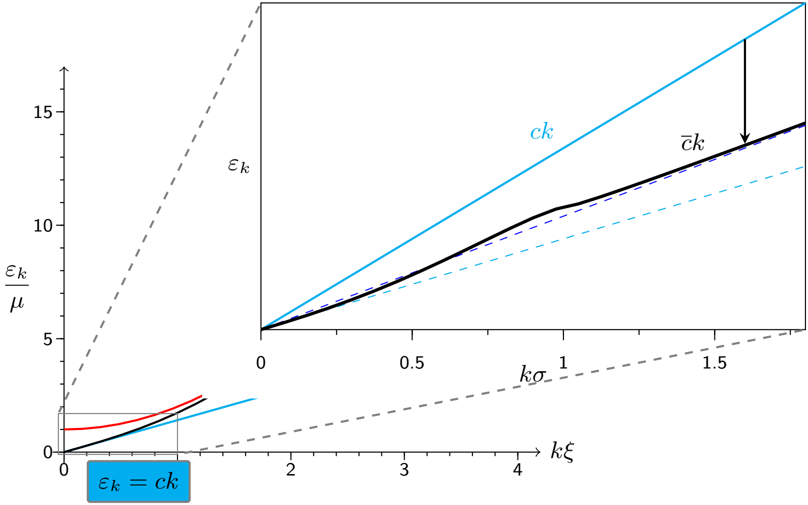
- Renormalized dispersion relation $\hbar\omega = \varepsilon_k + \text{Re}\Sigma$
- Finite mean free path $(kl_\text{s})^{-1} \propto |\text{Im}\Sigma|$

Mean free path l_s

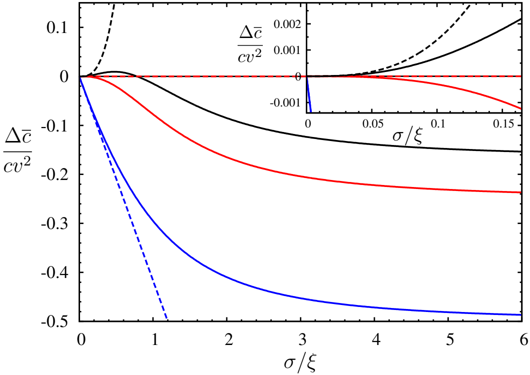


- Diverging mean free path for both low and high energy excitations

Renormalized speed of sound



Renormalized speed of sound ($k \to 0$)



$\frac{\Delta c}{c}$	$d=1$	$d=2$	$d=3$
$\sigma \gg \xi$	$-\frac{1}{2}v^2$	$-\frac{1}{4}v^2$	$-\frac{1}{6}v^2$
$\sigma \ll \xi$	$-\frac{3}{16\sqrt{2}}r_1$	0	$+\frac{5}{48\sqrt{2}\pi}r_3$

$$v^2 = \lim_{\boldsymbol{r}\to 0} \overline{V(\boldsymbol{r})V(0)}/\mu^2$$

$$r_d = \lim_{\boldsymbol{k}\to 0} \overline{|V_{\boldsymbol{k}}|^2}/\mu^2 \propto v^2 \frac{\sigma^d}{\xi^d}$$

Condensate depletion due to disorder

Non-condensed density $\delta n := L^{-d} \int \mathrm{d}^d r \langle \delta\hat{\Psi}(\boldsymbol{r})^\dagger \delta\hat{\Psi}(\boldsymbol{r}) \rangle$

- Express $\delta\hat{\Psi}(\boldsymbol{r})$ in terms of $\delta\hat{n}_{\boldsymbol{k}}$ and $\delta\hat{\varphi}_{\boldsymbol{k}}$, then in terms of $\hat{\gamma}_{\boldsymbol{k}'}$ and $\hat{\gamma}^\dagger_{\boldsymbol{k}'}$
- Then [with $\check{n}(\boldsymbol{r}) = 1/n(\boldsymbol{r})$]

$$\delta n = \frac{1}{4nL^d} \sum_{\boldsymbol{k},\boldsymbol{k}'} \left\{ \left[a_k a_{k'} \check{n}_{\boldsymbol{k}'-\boldsymbol{k}} + \frac{n_{\boldsymbol{k}'-\boldsymbol{k}}}{a_k a_{k'}} \right] \left\langle \hat{\gamma}_{\boldsymbol{k}} \hat{\gamma}^\dagger_{\boldsymbol{k}'} + \hat{\gamma}^\dagger_{\boldsymbol{k}'} \hat{\gamma}_{\boldsymbol{k}} \right\rangle \right.$$

$$\left. + \left[a_k a_{k'} \check{n}_{\boldsymbol{k}'-\boldsymbol{k}} - \frac{n_{\boldsymbol{k}'-\boldsymbol{k}}}{a_k a_{k'}} \right] \left\langle \hat{\gamma}_{\boldsymbol{k}} \hat{\gamma}_{-\boldsymbol{k}'} + \hat{\gamma}^\dagger_{\boldsymbol{k}'} \hat{\gamma}^\dagger_{-\boldsymbol{k}} \right\rangle - 2\delta_{\boldsymbol{k}\boldsymbol{k}'} \right\}, \; a_k = \sqrt{\frac{\varepsilon^0_k}{\varepsilon_k}}$$

- Compute $n_{\boldsymbol{q}}$ and $\check{n}_{\boldsymbol{q}}$ (perturbatively) from Gross-Pitaevskii equation
- Compute $\left\langle \hat{\gamma}^\dagger_{\boldsymbol{k}'} \hat{\gamma}_{\boldsymbol{k}} \right\rangle$ etc.

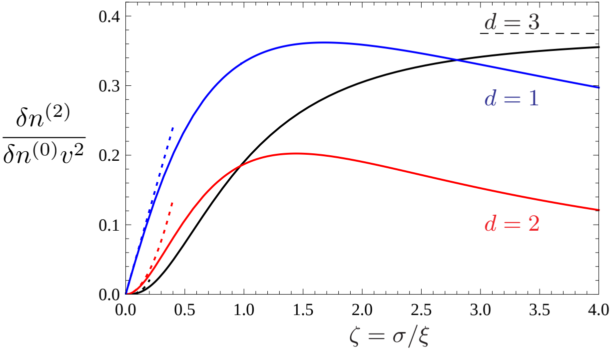
$$\left\langle \hat{\gamma}^\dagger_{\boldsymbol{k}'} \hat{\gamma}_{\boldsymbol{k}} \right\rangle = -\lim_{\tau \to 0^-} \left[-\left\langle T_\tau \hat{\gamma}_{\boldsymbol{k}}(\tau) \hat{\gamma}^\dagger_{\boldsymbol{k}'}(0) \right\rangle \right]$$

using Matsubara-Green functions and standard techniques

- Disorder average

Condensate depletion due to disorder

$$\delta n^{(2)} = \int \frac{d^d q}{(2\pi)^d} \overline{|V_{\boldsymbol{q}}|^2} \int \frac{d^d k}{(2\pi)^d} F(k, q, \theta)$$

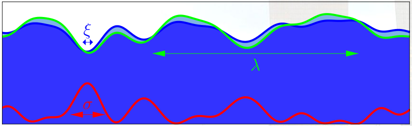


$\frac{\delta n^{(2)}(\zeta)}{\delta n^{(0)}}$	$d = 1$	$d = 2$	$d = 3$
$\sigma \gg \xi$	$-\frac{1}{8} v^2$	0	$\frac{3}{8} v^2$
$\sigma \ll \xi$	$0.235\, r_1$	$0.135\, r_2$	$0.160\, r_3$

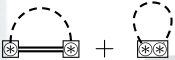
$$v^2 = \lim_{\boldsymbol{r} \to 0} \overline{V(\boldsymbol{r}) V(0)} / \mu^2$$

$$r_d = \lim_{\boldsymbol{k} \to 0} \overline{|V_{\boldsymbol{k}}|^2} / \mu^2 \propto v^2 \frac{\sigma^d}{\xi^d}$$

Conclusions



$$\hat{H} = \hat{H}_0 + \frac{1}{2}\hat{\Gamma}^\dagger \mathcal{V} \hat{\Gamma}$$

$\Sigma^{(2)} =$ 

Main results

- $kl_\mathrm{s} \gg 1$, speed of sound meaningful
- renormalization of speed of sound Δc
- true quantum depletion $\langle \delta\hat{\psi}^\dagger \delta\hat{\psi} \rangle_\mathrm{disorder}$
- arbitrary disorder correlation
- arbitrary dimension

Read more:

- CG & C.A. Müller, arxiv:1009.5448 (letter), arxiv:1101.4781 (regular article)

Thanks

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